THE AMALGAMATED FREE PRODUCT OF HYPERFINITE VON NEUMANN ALGEBRAS

A Dissertation

by

DANIEL ERNEST REDELMEIER

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2012

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ABSTRACT

The Amalgamated Free Product of Hyperfinite von Neumann Algebras. (May 2012)

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We examine the amalgamated free product of hyperfinite von Neumann algebras. First we describe the amalgamated free product of hyperfinite von Neumann algebras over finite dimensional subalgebras. In this case the result is always the direct sum of a hyperfinite von Neumann algebra and a finite number of interpolated free group factors. We then show that this class is closed under this type of amalgamated free product. After that we allow amalgamation over possibly infinite dimensional multimatrix subalgebras. In this case the product of two hyperfinite von Neumann algebras is the direct sum of a hyperfinite von Neumann algebra and a countable direct sum of interpolated free group factors. As before, we show that this class is closed under amalgamated free products over multimatrix algebras.

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Oh, and thanks Abner for the LaTeX template.

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CHAPTER I

INTRODUCTION

A. Introduction

Since their first appearance in [Voi95] and [Voi85], the (reduced) amalgamated free product of C* and von Neumann algebras have seen considerable use. For example, Popa used the amalgamated free product of von Neumann algebras in [Pop93] to construct subfactors with arbitrary allowable index.

Dykema describes the free product (with amalgamation over $\mathbb{C}$) of hyperfinite von Neumann algebras in [Dyk93]. In [Dyk95] he describes the amalgamated free product of multimatrix algebras (with amalgamation over a multimatrix subalgebra). Here we will extend these results to hyperfinite von Neumann algebras with amalgamation over multimatrix algebras (finite dimensional in Chapter IV and infinite dimensional in Chapter V).

In [Dyk11] Dykema describes the amalgamated free product of certain hyperfinite algebras over a finite dimensional subalgebra, and shows that a related class is closed under this kind of amalgamated free product. This result was used by Kodiyalam and Sunder in [KS11] and in the related paper by Guionnet, Jones, and Shlyakhtenko in [GJS11]. We extend this to a larger class in Chapter IV, and find a similar (but necessarily larger) class which is closed under amalgamation over infinite dimensional multimatrix algebras in Chapter V.

The journal model is Proceedings of the American Mathematical Society.

B. Notation

Throughout this document we will be working with finite von Neumann algebras with a specified normal faithful tracial state. In most cases the trace will be clear and just referred to as τ. In addition, in many cases we shall have a specific subalgebra D and trace preserving conditional expectation E_D onto D. When we refer to the expectation without specifying otherwise, this is the one we are referring to.

We will also often write von Neumann algebras as the direct sums of subalgebras. In this case we use the following notation:

$$A = \overset{p_1}{A_1} \oplus \overset{p_2}{A_2} \oplus \ldots.$$

Here p_i is used to denote the projection which is the central support of A_i in A (also referred to as the matrix unit of A_i). In addition if A_i is a matrix algebra, then we may use the notation $\underset{t_i}{A_i}$ to denote the trace of a minimal projection in A_i is t_i.

In most cases we will be working with von Neumann algebras that can be written in the following format:

$$H \oplus \bigoplus_{i \in I} F_i \oplus \bigoplus_{j \in J} M_{n_j},$$

where H is a diffuse hyperfinite algebra, the F_i are interpolated free group factors, and the M_{n_j} are matrix algebras. We refer to a von Neumann algebra which is the (possibly countably infinite) direct sum of matrix algebras as a *multimatrix algebra*. We write $vN(X)$ to denote the von Neumann algebra generated by a set X.

We use $\Lambda(X, Y)$ to denote the set of all non-empty alternating words of elements of X and Y. For an algebra X with a trace, we use X^0 to denote the elements of X with trace zero and if it has an expectation E_D^X onto a subalgebra D, we use X^{00} to denote elements of X with expectation zero. Note, since the expectation is trace preserving, $X^0 \subseteq X^{00}$.

CHAPTER II

BACKGROUND

A. The Amalgamated Free Product

The amalgamated free product of von Neumann algebras, introduced by Voiculescu in [Voi95] and [Voi85], is an extension of the standard free product (which we can think of as amalgamated over $\mathbb{C}$). The basic objects in this construction are finite von Neumann algebras, with specified trace, and with trace preserving conditional expectations onto a unital subalgebra D.

Definition 1. Let A be a von Neumann algebra with trace τ and trace preserving conditional expectation E_D^A onto a unital subalgebra D. Subalgebras A_1 and A_2, both containing D, are said to be *free with amalgamation over* D if for any $a_i \in A_1^{00} \cup A_2^{00}$, so that if $a_i \in A_k$ then $a_{i+1} \notin A_k$, then $E_D^A(a_1 a_2 \dots a_n) = 0$.

The *amalgamated free product* of two finite von Neumann algebras A, B, each with specified normal faithful tracial states and trace preserving conditional expectations onto unital subalgebra D is a von Neumann algebra generated by subalgebras A and B, with trace and conditional expectation onto D agreeing with those of A and B in which A and B are free with amalgamation over D.

In order to construct this, we first construct the reduced amalgamated free product for C*-algebras. Start with C*-algebras $\{A_i\}_{i\in I}$, each with non-degenerate conditional expectation E_D^i onto unital subalgebra D. We then take their GNS representations $(\pi_i, \mathcal{H}_i, \zeta_i)$, where $\mathcal{H}_i$ is the Hilbert D-bimodule obtained from (A_i, E_D^i) by separation and completion, π_i is the representation of $A_i \to \mathcal{B}(\mathcal{H}_i)$, and $\zeta_i = \hat{1}$ (the element in $\mathcal{H}_i$ corresponding to $1 \in A_i$).

The expectation E_D^i can be extended to an orthogonal projection $E_D^{\mathcal{H}_i} : \mathcal{H}_i \to$

$\zeta_i D$. Define $\mathcal{H}_i^{00}$ to be the kernel of $E_D^{\mathcal{H}_i}$, and note $\mathcal{H}_i \cong \mathcal{H}_i^{00} \oplus \zeta_i D$. We then construct a Hilbert D-bimodule as follows:

$$\mathcal{H} = \zeta D \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{(i_1,\ldots,i_n) \in I^n \\ i_k \neq i_{k+1}}} \mathcal{H}_{i_1}^{00} \otimes_D \mathcal{H}_{i_2}^{00} \otimes_D \ldots_D \mathcal{H}_{i_n}^{00}.$$

The *left regular representation*, λ_i, of A_i on $\mathcal{H}$ is defined through the following cases:

1. For $a \in D \subseteq A_i$ and $h \in \mathcal{H}$, $\lambda_i(a)h = ah$, using the left action of D on $\mathcal{H}$.

2. For $a \in A_i^{00}$, and $h \in \zeta D$, $\lambda_i(a)h = \hat{a}h = \pi_i(a)h \in \mathcal{H}_i^{00}$, using the right action of D on $\mathcal{H}_i^{00}$.

3. For $a \in A_i^{00}$, and $h \in \mathcal{H}_{i_1}^{00} \otimes_D \ldots \mathcal{H}_{i_n}^{00}$, $i_1 \neq i$, $\lambda_i(a)h = \hat{a} \otimes_D h \in \mathcal{H}_i^{00} \otimes_D \mathcal{H}_{i_1}^{00} \otimes_D \cdots \otimes_D \mathcal{H}_{i_n}^{00}$

4. For $a \in A_i^{00}$ and $h = h_i \otimes h' \in \mathcal{H}_i^{00} \otimes_D \mathcal{H}_{i_2^{00}} \otimes_D \cdots \otimes_D \mathcal{H}_{i_n}^{00}$, then $\lambda_i(a)h = (\pi_i(a)h_i - E_D^i(\pi_i(a)h_i)) \otimes_D h' + E_D^i(\pi_i(a)h)h'$.

Extending linearly determines the representation. We are now ready to define the reduced amalgamated free product for C*-algebras.

Definition 2. The *reduced amalgamated free product* of $\{A_i, E_D^i\}_{i \in I}$, $(*_D A_i)$ is the C*-algebra generated by the left regular representations of all the A_i and equipped with the conditional expectation onto D, $E_D(x) = \langle \zeta, x\zeta \rangle$.

Now we check this has the desired properties. First, we show that λ_i is injective, implying $\lambda_i(A_i)$ is a copy of A_i embedded in the reduced amalgamated free product. Since π_i is injective (since E_D^i is non-degenerate), and if we restrict to the subspace $\zeta D \oplus \mathcal{H}_i^{00} \cong \mathcal{H}_i$ then $\lambda_i(a)$ acts as $\pi_i(a)$ for all $a \in A_i$ we see that λ_i is injective. By definition, the $\lambda(A_i)$ generate $(*_D A_i)$. Next note for $a \in A_i$

$$E_D(a) = \langle \zeta, \lambda_i(a)\zeta \rangle = \left\langle \zeta, \lambda_i(a - E_D^i(a))\zeta \right\rangle + \left\langle \zeta, \lambda_i(E_D^i(a))\zeta \right\rangle$$

$$= \left\langle \zeta, E_D^i(a)\zeta \right\rangle = E_D^i(a),$$

and thus the expectations are compatible. Finally we check that the algebras are free with amalgamation over D. Take, $a_1, \ldots, a_n$, with $a_k \in A_{i_k}^{00}$ and $i_k \neq i_{k+1}$ for any k. Then when calculating $\lambda_{i_1}(a_1)\lambda_{i_2}(a_2)\ldots\lambda_{i_n}(a_n)\zeta$, we first apply the second rule in the definition of the left regular representation, which takes $\zeta \in D$ to $\mathcal{H}_{i_n}^{00}$, then in all subsequent steps we apply the third (since $i_k \neq i_{k+1}$), giving us a result in $x \in \mathcal{H}_{i_1}^{00} \otimes_D \cdots \otimes_D \mathcal{H}_{i_n}^{00}$, and thus $E_D(a_1 \ldots a_n) = \langle \zeta, x \rangle = 0$.

As with ordinary free products, we can use the definition of freeness with amalgamation to determine $E_D(a_1 a_2 \ldots a_n)$ in $(*_D A_i)$ with $a_k \in A_{i_k}$, in terms of the expectations $E_D^{i_k}$. Just like in the ordinary free product, we write each a_i as $(a_i - E_D^i(a_i)) + E_D^i(a_i)$, expand by linearity, and repeat as necessary. However the computation is often significantly more complicated, since D may not commute with the A_i.

We are now ready to use this to define the amalgamated free product for von Neumann algebras, as in [Pop93] by Popa. Take finite von Neumann algebras $\{A_i\}_{i\in I}$, each equipped with both a faithful normal tracial state τ_i, and trace preserving conditional expectation E_D^i onto a unital subalgebra D (equipped with the trace τ_D). Let A be the C^* reduced amalgamated free product of these von Neumann algebras.

Equip A with a trace simply by defining $\tau(a) = \tau_D(E_D(a))$. We need to check that this is a trace on A. It is clear that it is still a trace on D, so we need only check elements of the form $a = a_1 \ldots a_n$ where $a_k \in A_{i_k}^{00}$, and $i_k \neq i_{k+1}$ for all k. In particular we need to show that for a as before, and $b = b_1 \ldots b_m$ with $b_k \in A_{j_k}^{00}$ then $\tau(ab) = \tau(ba)$. This is zero unless $m = n$ and $i_k = j_{n-k+1}$ for all k. In this case, note that $E(ab) = E(a_1 E(a_2 E(\ldots E(a_n b_1) b_2) \ldots) b_n)$. Noting $a_1 E(a_2 E(\ldots E(a_n b_1) b_2) \ldots) b_n \in$

A_{i_1}, we see

$$\tau(E(ab)) = \tau_{i_1}\left(a_1 E\left(a_2 E\left(\ldots E\left(a_n b_1\right) b_2\right) \ldots b_{n-1}\right) b_n\right)$$

$$= \tau_{i_1}\left(b_n a_1 E\left(a_2 E\left(\ldots E\left(a_n b_1\right) b_2\right) \ldots b_{n-1}\right)\right)$$

$$= \tau\left(E\left(b_n a_1 E\left(a_2 E\left(\ldots E\left(a_n b_1\right) b_2\right) \ldots b_{n-1}\right)\right)\right)$$

$$= \tau(E(b_n a_1) E(a_2 E(\ldots E(a_n b_1) b_2) \cdots_{n-1}))$$

$$= \tau(E(E(b_n a_1) a_2 E(\ldots E(a_n b_1) b_2) \ldots b_{n-1})).$$

Repeating this process we get the desired result. Since E_D is a positive map, it follows that for any $a \in A$, $a > 0$, then $E_D(a) > 0$ and thus $\tau(a) = \tau_D(E_D(A)) > 0$, so τ is faithful.

Definition 3. Let $\{A_i\}_{i \in I}$ be finite von Neumann algebras equipped with normal faithful tracial states τ_i and trace preserving conditional expectations E_D^i onto unital subalgebra D. Let A be the reduced amalgamated free product of these algebras over D as C*-algebras, equipped with the trace τ as defined above. Then the *amalgamated free product* over D is the double commutant of the GNS representation of A with respect to τ.

Lemma 4. *Let A and B be subalgebras of a von Neumann algebra $\mathcal{M}$ with common subalgebra D. If all alternating words in the expectationless elements of A and B have trace zero then A and B are free with amalgamation over D (i.e. these words are also expectationless).*

Proof. Assume for contradiction there is such a word w with non-zero expectation d. Then note that wd^* is also such a word since an expectationless element times an element of D is still expectationless. However $E_D(wd^*) = E(w)d^* = dd^*$, which by the faithfulness of the trace is non-zero. Thus contradicting our assumption. $\square$

B. Interpolated Free Group Factors

Interpolated free group factors, independently introduced by Dykema in [Dyk94] and Rădulescu in [Răd94], generalise the standard free group factors $L(F_n)$ for $n = 2, 3, \ldots, \infty$. We will follow the construction from [Dyk94].

Definition 5. Let R be the hyperfinite II_1 factor, and $\omega = \{X_t, t \in T\}$ be a semicircular system, such that R and ω are contained in a von Neumann algebra with faithful normal trace τ, where R and ω are free. Let $1 < r \leq \infty$, and choose $\{p_t\}$ projections in R such that $r = 1 + \sum_{t \in T} \tau(p_t)^2$. Then the *interpolated free group factor*, $L(F_r)$ is the von Neumann algebra generated by $R \cup \{p_t X_t p_t | t \in T\}$.

Before we can prove that this is well defined, we need a few tools.

Lemma 6. *For A, B, and C finite von Neumann algebras (with specified traces), then for $n \geq 1$, consider $\mathcal{N} = (M_n \oplus B) * C$ and $\mathcal{M} = ((A \otimes M_n) \oplus B) * C$, with the natural inclusion. For any minimal projection of M_n, $p\mathcal{M}p \cong C * p\mathcal{N}p$, where C on the right side is $C \otimes p$. Moreover the central supports of p in $\mathcal{M}$ and $\mathcal{N}$ are the same.*

This was proved in [Dyk93] as Theorem 1.2, and we prove a more general version as Lemma 43 later.

Proposition 7. *Let (A, τ) be a space, containing a finite sequence $a_1, \ldots, a_n$ and a projection p free from these with trace λ,*

$$\kappa'_n(pa_1p, \ldots, pa_np) = \lambda^{n-1}\kappa_n(a_1, \ldots, a_n)$$

where κ' is the cumulant in pAp.

Proof. We use τ' to denote the renormalized trace in pAp. We will also use the notation $\tau_\pi(x_1, \ldots, x_m)$ for a non-crossing partition π of m to denote $\prod_{X \in \pi} \tau(\prod_{i \in X} x_i)$, where the inside product is taken in order of increasing index.

First examine $\tau'_n(pa_1p, \ldots, pa_np)$. Since p is free from $\{a_i\}_{i=1}^n$ we can expand this trace, and write it in terms only of λ and the $\tau_\pi(a_1, \ldots, a_n)$ for all $\pi \in NC(n)$. It can be expanded as

$$\tau'_n(pa_1p, \ldots, pa_np) = \sum_{\pi \in NC(n)} d_{\pi,1_n} \tau_\pi(a_1, \ldots, a_n),$$

for some set of coefficients $d_{\pi,1_n}$ determined only by π and λ. Naturally we can extend this to:

$$\tau'_\sigma(pa_1p, \ldots, pa_np) = \sum_{\pi \leq \sigma} d_{\pi,\sigma} \tau_\pi(a_1, \ldots, a_n)$$

where $d_{\pi,\sigma}$ is again determined only by σ, π, and λ.

We then claim that for $\psi \leq \sigma \in NC(n)$,

$$\sum_{\pi, \psi \leq \pi \leq \sigma} d_{\pi,\sigma} = \lambda^{n-|\psi|}.$$

Let $m = |\psi|$. Since ψ is a non-crossing partition we can choose an order of the parts of ψ, $X_1, \ldots, X_m$, so that each X_i is a block if $X_1, \ldots, X_{i-1}$ are removed. Let $n_i = |X_i|$. Define b_i for $1 \leq i \leq m$ to be a generator for the group algebra of $\mathbb{Z}_{n_i}$, and so that $\{b_1, \ldots, b_m\}$ are mutually free. Let $a_j = b_i$ for all j in the ith part of ψ.

From our definition of $d_{\pi,\sigma}$ we know

$$\tau'_\sigma(pa_1p, \ldots, pa_np) = \sum_{\pi \leq \sigma} d_{\pi,\sigma} \tau_\pi(a_1, \ldots, a_n).$$

Note that $\tau(b_i^k)$ is zero if $0 < k < n_i$ and $b_i^{n_i} = I$. Since we have only n_i of each b_i these are the only possible cases.

Examining the right side, consider any part Y of π. First check if it has any elements in common with X_1, which then correspond to b_1s in $\tau_Y(a_1, \ldots, a_n)$. If it has any they must be in a block. If it does not have exactly n_1 or zero elements of X_1 then we have an element of trace zero which is free from the rest of the product and thus the trace is zero. If it does, then the n_1 or zero copies of b_1 multiply to the identity. We can then repeat this process for the rest of the X_i and the rest of the parts of π. Thus the trace is zero if and only if each part of π contains only entire parts of ψ, which is exactly if $\pi \geq \psi$. So we see

$$\tau_\pi(A) = \begin{cases} 1 & \pi \geq \psi \\ 0 & \text{otherwise} \end{cases}.$$

Thus the right side of the equation is

$$\sum_{\psi \leq \pi \leq \sigma} d_{\pi,\sigma}.$$

On the left side, begin with X_1, the first part of ψ. We know X_1 is in a block, so there are $n_1 - 1$ ps placed between the b_1s corresponding to this block, which we can expand as $(p-\lambda)+\lambda$. We find that in any of the terms of the expansion where we retain a $(p-\lambda)$ it then separates the b_1s in such a way that neither side can ever have a non-zero trace, and thus the term has trace zero. Thus we are left with the term where all those ps are replaced by λ. This means the b_1s cancel forming the identity, and we can factor out λ^{n_1-1}. We are thus left with λ^{n_1-1} times our original trace with the first part of ψ removed. We can repeat this process with each part in order. This gives us

$$\tau'_\sigma(pa_1p, \ldots, pa_np) = \tau'(p)^{|\sigma|} \prod_{i=1}^{m} \lambda^{n_i-1} = \lambda^{n-m},$$

proving the claim.

Then, by definition,

$$\kappa_n'(pa_1p\ldots,pa_np)=\sum_{\sigma\in NC(n)}\tau_\sigma'(pa_1p.\ldots,pa_np)\mu(\sigma,1_n).$$

By our choices of $d_{\pi,\sigma}$,

$$=\sum_{\sigma\in NC(n)}\sum_{\pi\leq\sigma}d_{\pi,\sigma}\tau_\pi(a_1,\ldots,a_n)\mu(\sigma,1_n),$$

and by the moment cumulant formula

$$=\sum_{\sigma\in NC(n)}\sum_{\pi\leq\sigma}d_{\pi,\sigma}\sum_{\psi\leq\pi}\kappa_\psi(a_1,\ldots,a_n)\mu(\sigma,1_n).$$

Reordering the sum we see

$$=\sum_{\psi\in NC(n)}\kappa_\psi(a_1,\ldots,a_n)\sum_{\sigma\geq\psi}\mu(\sigma,1_n)\sum_{\pi,\psi\leq\pi\leq\sigma}d_{\pi,\sigma}.$$

By our claim we get

$$=\sum_{\psi\in NC(n)}\kappa_\psi(a_1,\ldots,a_n)\sum_{\sigma\geq\psi}\mu(\sigma,1_n)\lambda^{n-|\psi|}.$$

Noting that $\lambda^{n-|\psi|}$ does not depend on σ, and by the basic properties of the möbius function, the inner sum is zero unless $\psi=1_n$. Thus we are left with

$$=\lambda^{n-1}\kappa_n(a_1,\ldots,a_n).$$

$\square$

Proposition 8. *In (A,τ), let $\{e_{ij}\}_{j=1}^n$ be a set of matrix units in A. Take $\{a^{(k)}\}_{k=1}^m$ in A free from $\{e_{ij}\}_{j=1}^n$. Let $a_{ij}^{(k)}=e_{1i}a^{(k)}e_{j1}$. Then:*

$$\kappa_N^{e_{11}Ae_{11}}(a_{i_1i_2}^{(k_1)},a_{i_2i_3}^{(k_2)},\ldots,a_{i_Ni_1}^{(k_N)})=\frac{1}{n^{N-1}}\kappa_N^A(a^{(k_1)},a^{(k_2)},\ldots,a_k^{(k_N)}),$$

for $1\leq i_j\leq n$ and $1\leq k_j\leq m$. The cumulant is zero if anywhere we have $a_{ij}^{(k)}$

followed by $a^{(k')}_{i'j'}$ *with* $j \neq i'$.

The proof of this is similar to the previous proposition, and can be found in [NS06] as Theorem 14.18. The following theorem is similar to Theorem 14.20 in the same book:

Proposition 9. *For* (A, τ), *let* $\{a_{ij}\}_{1\leq i,j\leq n} \subset A$, *and* B *a subalgebra of* A. *Then the following are equivalent.*

1. *The algebra* $M_n \otimes B$ *and* X *the matrix formed by the elements of* $\{a_{ij}\}_{1\leq i,j\leq n}$ *are free in* $M_n \otimes A$.
2. *Only cyclic cumulants (i.e. of the form* $\kappa_m(a_{i_1i_2}, a_{i_2i_3}, \ldots, a_{i_mi_1})$*) are non-zero, and those depend only on their length, and* B *is free from* $\{a_{ij}\}_{1\leq i,j\leq n}$.

Proof. To show the first implies the second, note that Proposition 8 gives us the property of the cumulants, by setting $X = a^{(1)}$, and the e_{ij} to be the matrix units from $M_n(B)$. To prove that B is free from $\{a_{ij}\}_{1\leq i,j\leq n}$, we apply Lemma 6. This shows us that

$$e_{11}(B \otimes M_n(\mathbb{C}) * vN(\{a_{ij}\}_{1\leq i,j\}}))e_{11} \cong B * e_{11}(M_n(\mathbb{C}) * vN(\{a_{ij}\}_{1\leq i,j\}}))e_{11},$$

in the natural way.

For the other direction, note that the joint moments of X and $M_n(B)$ are entirely determined by the joint moments of $\{a_{ij}\}_{\leq i,j\leq n}$ and B, which are then determined by their cumulants, which we have determined. So we need only show that the cumulants arise from an X and $M_n(B)$ which are free.

We define X' to be a random variable with cumulants $\kappa_m(X') = n^{m-1}\kappa_m(a_{ii})$ free from $M_n(B)$. Then using the first direction, we see this has the desired cumulants, and thus X is free from $M_n(B)$. $\square$

Lemma 10. *Let R be the hyperfinite II_1 factor, and let $\{X_t\}_{t=0}^{n^2}$ be a semicircular system free from it, and let $p \in R$ be a projection with trace $1/n$. Let $A = vN(R \cup \{X_0\})$ and $B = vN(R \cup \{pX_tp\}_{t=1}^{n^2})$, then $A \cong B$.*

Proof. Start by considering $\mathcal{M} = vN(R' \cup \{a_{ij}\}_{1 \leq i,j \leq n})$ where $\{a_{ii}\}_{i-1}^{n}$ are semicircular random variables, and $\{a_{ij}\}_{1 \leq i < j \leq n}$ are circular random variables, all free from each other and R, and $a_{ji} = a_{ij}^*$. It is easy to see $\mathcal{M}$ is isomorphic to pBp, and consequently $vN(R' \otimes M_n \cup \{X\})$ where X is the matrix formed from the $\{a_{ij}\}$, is isomorphic to B.

Next we show that cumulants on $\{a_{ij}\}_{1 \leq i,j \leq n}$ are only non-zero if they are cyclic (i.e. of the form $\kappa_m(a_{i(1)i(2)}, a_{i(2)i(3)}, \ldots, a_{i(m)i(1)})$), and that this value only depends on the length of the cumulant (m).

In fact since mixed cumulants vanish, the only cumulants that may not vanish are those composed of only of $\{a_{ij}, a_{ji}\}$ for specific (possibly identical) i, j. If $i \neq j$, then these are circular, and the only non-vanishing case is $\kappa_2(a_{ij}, a_{ji}) = \kappa_2(a_{ji}, a_{ij}) = 1$, which are both cyclic. If $i = j$ then the only non-vanishing case is $\kappa_2(a_{ii}, a_{ii}) = 1$, which is cyclic, and of the same value.

Then by Proposition 9, in $B \cong \{R' \otimes M_n \cup X\}$, where X is the matrix formed from the $\{a_{ij}\}$, X is free from $R \otimes M_n$. Now we need only show that X_t is semicircular. It is clear that it is self adjoint, so we need only check the cumulants of the form $\kappa_m(X)$. However from Proposition 7, we see $\kappa_m(X) = \frac{1}{n^{m-1}} \kappa_m^{pXp}(pAp) = \frac{1}{n^{m-1}} \kappa_m(a_{ii})$. Thus this is the only non-vanishing cumulant of X is $K_2(X) = \frac{1}{n}$. Thus X is semicircular and $B \cong A$.

□

We are now ready to prove that interpolated free group factors are well defined, with he following proposition, proved in [Dyk94]:

Proposition 11. *For $A = vN(R \cup p_t X_t p_t, t \in T\})$ and $B = vN(R \cup \{q_s X_s q_s, s \in S\})$, in the format above, if $\sum_{t\in T} \tau(p_t)^2 = \sum_{s \in S} \tau(p_s)^2$, then $A \cong B$.*

Proof. To prove this, we will show that any such algebra is isomorphic to one in standard form. First choose a pairwise orthogonal set of projections $\{f_k\}_{k=1}^{\infty}$ in R where $\tau(f_k) = 2^{-k}$ and set f_0 to be the identity. Let N_ℓ be the ℓth term in the base 4 expansion of $r - 1 = \sum_{t \in T} \tau(p_t)^2$ for $\ell \geq 1$ and $N_0 = \lfloor r - 1 \rfloor$. Choose an index set $S \subseteq \mathbb{N}^2$ where $(a, b) \in \mathbb{N}$ if $a \geq 0$ and $1 \leq b \leq N_a$. Then this is the standard form and we we will prove $A \cong vN(R \cup \{f_a X_{(a,b)} f_a, (a, b) \in S\})$.

First we break up each p_t in terms of the f_a. We use unitaries in R to conjugate each p_t so that $p_t = \sum_{k \in K} f_k$ (use the binary expansion of $\tau(p_t)$), for any $p_t \neq 1$, and note if $p_t = 1$ then $p_t = f_0$. First assume $\tau(p_t)$ is a dyadic rational $\frac{m}{2^n}$. Then applying Lemma 10, we see $vN(R \cup \{p_t X_t p_t\}) \cong vN(R \cup \{f_n X_k f_n\}_{k \in K})$ where K is of size m^2. Note it is easy to check that $\sum_{k \in K} \tau(f_n)^2 = m^2 \frac{1}{4^n} = \tau(p_t)^2$.

Now using Lemma 10 again, we get the following equation:

$$vN(R \cup \{f_k X_{t_i} f_k, i \in \{1, \ldots, 4\}\}) \cong vN(R \cup \{f_{k-1} X_t f_{k-1}\}\}). \tag{2.1}$$

Note here again, $\tau(f_{k-1})^2 = 2^{-2k+2}$, $\sum_{i=1}^{4} \tau(f_k)^2 = 4(2^{-2k}) = 2^{-2k+2}$.

We can repeatedly apply this equation to $vN(R \cup \{f_n X_k f_k\}_{k \in K})$, to give us an isomorphic algebra of the form $vN(R \cup \{f_{n_k} X_k f_{n_k}, k \in K'\})$, where for any n, there are at most three $k \in K'$ such that $n_k = n$. Since at each stage we preserve the sum of the squares of the traces, the number of $k \in K'$ such that $n_k = n$ is equal to the nth digit of the base 4 expansion of $\tau(p_t)$. Taking the inductive limit, we can extend this from dyadic rationals to any trace.

Now applying this inductively to each $t \in T$, we get that

$$A \cong vN(R \cup \{f_{k_s} X_s f_{k_s}, s \in S\}),$$

where, using equation 2.1 as necessary, we ensure that for any k there are at most three s such that $k = k_s$. Since each step preserves the sum of the squares of the traces, then $\sum_{s\in S}\tau(f_{k_s}) = r-1$ and thus this is the standard form. $\square$

The following theorem, proved in both [Dyk94] and [Răd94], gives the key properties of the interpolated free group factors.

Theorem 12. *The interpolated free group factors satisfy the following properties:*

1. *If $r \in \mathbb{Z}, 2 \le r \le \infty$ then $L(F_r)$ is the factor associated to the free group on r elements.*
2. *For $1 < r, r' \le \infty$, $L(F_r) * L(F_{r'}) = L(F_{r+r'})$.*
3. *For $1 < r \le \infty$ and $0 < \gamma < \infty$, $L(F_r)_\gamma = L(F_{1+(r-1)/\gamma^2})$, where $L(F_r)_\gamma$ is the compression or dilation of $L(F_r)$ by γ.*

Proof. Property 1 is clear from the choice of $p_t = 1$ for $r-1$ projections, giving us the von Neumann algebra generated by $R \cup \{X_t\}_{t=2}^r$, with each free, which is then $L(F_r)$.

For property 2, consider $L(F_r) \cong vN(R \cup \{p_tX_tp_t\}_{t\in T})$ with $\sum_{t\in T}\tau(p_t)^2 = r-1$ and $L(F_{r'}) \cong vN(R' \cup \{q_tX_tq_t\}_{t\in T'})$ with $\sum_{t\in T'}\tau(q_t)^2 = r'-1$ (for T and T' disjoint). Then $L(F_r) * L(F_{r'}) \cong vN(R \cup R' \cup \{p_tX_tp_t\}_{t\in T} \cup \{q_tX_tq_t\}_{t\in T'})$. Now since $R * R' \cong R * L(\mathbb{Z})$, where the isomorphism fixes the first copy of R, we can find a semicircular element X_0 which generates $L(\mathbb{Z})$, and unitaries $U_{t'} \in R * R'$ such that $U_tq_tU_t^* = p_{t'} \in R$ for $t \in T'$. Then, setting $p_0 = 1$, we see

$$L(F_r) * L(F_{r'}) \cong vN(R \cup \{p_tX_tp_t\}_{t\in T\cup T'\cup\{0\}}).$$

Noting $\sum_{t\in T\cup T'\cup\{0\}} \tau(p_t)^2 = r - 1 + r' - 1 + 1 = r + r' - 1$, thus this is isomorphic to $L(F_{r+r'})$.

For property 3, we can assume that $0 < \gamma < 1$. Let $L(F_r) = vN(R \cup \{p_t X_t p_t\}_{t\in T})$, as usual with $\sum_{t\in T} \tau(p_t)^2 = r - 1$. Then take projection $p \in R$ with $\tau(p) = \gamma$. Without loss of generality we assume $p_t \leq p$ for all $t \in T$ (we may have to modify our choice of p_t for this). Then we see $pL(F_r)p \cong vN(pRp \cup \{p_t X_t p_t\}_{t\in T})$. Under the new normalised trace, $\tau_P = \frac{1}{\gamma}\tau$, we see that $\sum_{t\in T} \tau_P(p_t)^2 = \frac{r-1}{\gamma^2}$. Since pRp is still a hyperfinite II_1 factor, and it can be checked that these are all still free (as in Theorem 1.3 in [Dyk94]), thus this is isomorphic to $L(F_{1+(r-1)/\gamma^2})$ as desired.

$\square$

It remains an open question whether the free group factors are isomorphic or not. As proved in [Dyk94] and [Rǎd94], either all interpolated free group factors $L(F_r)$ for $r > 1$ are isomorphic or they are all non-isomorphic.

C. Free Dimension

The concept of *free dimension* was introduced by Dykema in [Dyk93]. Free dimension applies in particular to the type of von Neumann algebra we will be dealing with here, and will be useful in a number of our calculations.

Definition 13. Let A be a finite von Neumann algebra with specified normal faithful tracial state which is of the format

$$A = H \oplus \bigoplus_{i\in I} \overset{p_i}{L(F_{r_i})} \oplus \bigoplus_{j\in J} \underset{t_j}{M_{n_j}},$$

where H is a diffuse hyperfinite algebra, the $L(F_{r_i})$ are interpolated free group factors and the M_{n_j} are matrix algebras. The *free dimension* of A (denote $\mathrm{fdim}(A)$) is equal to

$$1 + \left(\sum_{i \in I} \tau(p_i)^2 (r_i - 1) \right) - \sum_{j \in J} t_j^2.$$

From this we observe the following properties:

1. For A a diffuse hyperfinite algebra, $\mathrm{fdim}(A) = 1$.

2. For $A = L(F_r)$, an interpolated free group factor, $\mathrm{fdim}(A) = r$.

3. For $A = \oplus_{j \in J} \underset{t_j}{M_{n_j}}$ a multimatrix algebra, $\mathrm{fdim}(A) = 1 - \sum_{j \in J} t_j^2$.

4. For $A = L(G)$, where G is an amenable discrete group, $\mathrm{fdim}(A) = 1 - |G|^{-1}$.

5. For any A, $\mathrm{fdim}(A) \geq 0$, and $\mathrm{fdim}(\mathbb{C}) = 0$.

6. For von Neumann algebras (A, τ_A) and (B, τ_B), with $\mathrm{fdim}(A) = a$ and $\mathrm{fdim}(B) = b$, then $(A \oplus B, \tau)$ where $\tau|_A = \lambda \tau_A$ and $\tau|_B = (1 - \lambda)\tau_b$ then $\mathrm{fdim}(A \oplus B) = 1 + \lambda^2(a - 1) + (1 - \lambda)^2(b - 1)$.

Unfortunately we do not know whether free dimension is always well defined, noting the second property above. Obviously if it turns out that all the free group factors are isomorphic, then this is not well defined. In that case the interpolated free group factors would have free dimension of all values greater than 1. In general we use the notion of free dimension to distinguish interpolated free group factors, so if they are all isomorphic we do not really need it. Another way to look at it is to think of free dimension with respect to generating sets rather than algebras, as was used in [Dyk11] and [Dyk02]. In this document when we write $\mathrm{fdim}(A) = r$ we mean "A has a generating set of free dimension r".

Definition 14. For a von Neumann algebra (A, τ) with free dimension, and a value $0 < t \leq 1$, we define the *free dimension contribution* of A with trace t as $t^2(\mathrm{fdim}(A) - 1)$. Denote this $\mathrm{fdimC}_t(A)$.

Note the following properties

1. If $A = \bigoplus_{i\in I} \overset{p_i}{A_i}$ then $\mathrm{fdim}(A) - 1 = \mathrm{fdimC}_1(A) = \sum_{i\in I} \mathrm{fdimC}_{\tau(p_i)}(A_i)$.
2. If p is a projection with trace t and F is an interpolated free group factor, then $\mathrm{fdimC}_t pFp = \mathrm{fdimC}_1 F$.
3. For any A, $\mathrm{fdimC}_t(A) \geq -t^2$, $\mathrm{fdimC}_t\mathbb{C} = -t^2$.
4. For any diffuse hyperfinite algebra A, $\mathrm{fdimC}_t A = 0$.
5. If $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$ then $\mathrm{fdimC}_t(\mathcal{M}) = \mathrm{fdimC}_t(A) + \mathrm{fdimC}_t(B) - \mathrm{fdimC}_t(D)$. (This will be useful since we will be proving conditions under which $\mathrm{fdim}(A *_D B) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$.
6. For projection $p \in A$ with central support q, $\mathrm{fdimC}_{\tau(p)}(pAp) = \mathrm{fdimC}_{\tau(q)}(qAq)$.

To prove the last property, using the first property it is sufficient to prove it when qAq is a factor or a diffuse hyperfinite algebra. If it is an interpolated free group factor, then this is property 2. If it is a matrix algebra, whose minimal projections have trace α in A, then $\mathrm{fdimC}_{\tau(p)}(pAp) = \mathrm{fdimC}_{\tau(q)}(qAq) = -\alpha^2$. If it is diffuse hyperfinite then $\mathrm{fdimC}_{\tau(p)}(pAp) = \mathrm{fdimC}_{\tau(q)}(qAq) = 0$.

D. Standard Embeddings

A key concept we will use later is the *standard embedding*, first defined by Dykema in [Dyk93].

Definition 15. Let $A = L(F_r)$ and $B = L(F_{r'})$ for $r < r'$ and let $\phi : A \to B$ be a unital embedding. We call ϕ a *standard embedding* if we can find a semicircular system $\omega = \{X_t\}_{t\in T}$ and a copy of R, the hyperfinite II_1 factor free from ω, and $p_t \in R$ so that

B is generated by R and $\{p_tX_tp_t\}_{t\in T}$, and such that if $T' = \{t \in T | p_tX_tp_t \in \phi(A)\}$ then $\phi(A) = vN(R \cup \{p_tX_tp_t\}_{t\in T'})$.

Example 16. If $L(F_n)$ and $L(F_m)$, $m > n$ are free group factors (with $m, n \in \mathbb{Z}$), then the embedding from $L(F_n) \to L(F_m)$ induced by the inclusion of groups $F_n \to F_m$ is a standard embedding. Here we know $L(F_n) = vN(R \cup \{X_2, \dots, X_n\})$ where the X_t correspond to Haar unitary generators for $L(F_n)$, representing generators for F_n. Thus the inclusion of $F_n \to F_m$ naturally gives us the inclusion into $L(F_m) = vN(R \cup \{X_2, \dots, X_m\}$.

A key property we use to prove an embedding is standard is the following proposition from [Dyk93].

Proposition 17. *Let A and B be interpolated free group factors, with fdim$(A) <$ fdim(B), with unital embedding $\phi : A \to B$. Let $p \in A$ be a projection. Then ϕ is standard if and only if $\phi|_{pAp} : pAp \to \phi(p)A\phi(p)$ is standard.*

Proof. First assume ϕ is standard, and $p \in A$. Since ϕ is standard we can find R, $\{X_t\}_{t\in T}$ a semicircular system free from it, and $p_t \in T$, and $T' \subset T$ such that and $B \cong vN(R \cup \{p_tX_tp_t, t \in T\})$ and $\phi(A) \cong vN(R \cup \{p_tX_tp_t, t \in T'\})$ under the same isomorphism. Since $p \in A$ we can choose this so that $p \in R$. As in the proof of Proposition 11, we can assume that $p_t \leq p$ for all t. Then we see $\phi(pAp) \cong vN(pRp \cup \{p_tX_tp_t, t \in T'\})$, and $pBp \cong vN(pRp \cup \{p_tX_tp_t, t \in T\})$, which is of the right form (as in the proof of the third property of Theorem 12). Thus $\phi|_{pAp}$ is standard.

Similarly if $\phi|_{pAp}$ is standard, we can represents pAp and pBp in the same way, and then extend R to generate A and B, showing ϕ is standard. $\square$

Another handy proposition from [Dyk93] is the following:

Proposition 18. *The inclusion* $L(F_r) \to L(F_r) * L(F_{r'})$, $1 < r \le \infty$, $1 \le r \le \infty$, *is standard.*

Proof. If F is an interpolated free group factor, this follows directly from the proof of property 2 of Theorem 12, and the definition of standard. In the case of $F = L(\mathbb{Z})$, this follows directly from the fact that $R * L(\mathbb{Z}) \cong vN(R \cup \{X_t\})$.

□

For our purposes, it will often be useful to work with a less restrictive definition, so we use:

Definition 19. A (not necessarily unital) embedding ϕ from an interpolated free group factor $A = L(F_r)$ into another interpolated free group factor $B = L(F_{r'})$ is called *substandard* if ϕ considered as an embedding into $\phi(I_A)B\phi(I_A)$ is either a standard embedding or an isomorphism.

Example 20. Let $A = L(F_2)$ and let $B = L(F_{7/4})$. By Theorem 12 $B \cong L(F_4) \otimes M_2 \cong (L(F_2) * L(F_2)) \otimes \mathcal{M}$. Then let $\phi : A \to B$ be the inclusion of F_2 into the upper left copy of $L(F_2) * L(F_2)$ in B. Then $\phi(I_A)B\phi(I_A) = e_{11}Be_{11} = L(F_2) * L(F_2)$. The restricted embedding is standard, by Proposition 18, and thus ϕ is substandard. Note this demonstrates that unlike a standard embedding, a substandard embedding can be an inclusion of an interpolated free group factor into another interpolated free group factor of smaller free dimension.

The following result, from [Dyk93] is the main reason the concept is useful:

Proposition 21. 1. *Let* A, B, C *be interpolated free group factors. Let* $\phi : A \to B$ *and* $\psi : B \to C$ *be substandard embeddings. Then* $\psi \circ \phi$ *is substandard, and if both* ϕ *and* ψ *are standard then it is too.*

2. *Let $(A_n, \tau_n) = L(F_{r_n})$ and let $\phi_n : A_n \to A_{n+1}$ be substandard embeddings. Let $\rho_n = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$. Then if $\lim_{n\to\infty} \tau_n(\rho_{n-1}(I_1)) > 0$, then the inductive limit of (A_n, ϕ_n) is $A = L(F_r)$ where $r = \lim_{n\to\infty} r_n$, and the embedding of A_n into A is substandard (if each ϕ_n was standard then it is standard).*

Proof. 1. First assume ϕ and ψ are both standard. In this case we choose two disjoint semicircular systems $\{X_t\}_{t\in T}$ and $\{X_s\}_{s\in S}$, and write $B \cong vN(R \cup \{p_s X_s p_s\}_{s\in S})$ with $\phi(A) = vN(R \cup \{p_s X_s p_s\}_{s\in S'})$ for some $S' \subset S$. Similarly we write $C \cong vN(R' \cup \{p_t X_t p_t\}_{t\in T})$ with $\psi(B) = vN(R' \cup \{p_t X_t p_t\}_{t\in T'})$ for some $T' \subset T$. However, this means that $vN(R \cup \{p_s X_s p_s\}_{s\in S}) \cong vN(R' \cup \{p_t X_t p_t\}_{t\in T'})$ under ψ. Thus $C = vN(R \cup \{p_s X_s p_s\}_{s\in S} \cup \{p_t X_t p_t\}_{t\in T\setminus T'})$. We use unitary conjugation to ensure the p_t are in R. Then the image under ϕ of A is $vN(R \cup \{p_s X_s p_s\}_{s\in S'})$ in B and thus applying ψ it is the same again, in C. Thus $\psi \circ \phi$ is standard.

If ϕ and ψ are only substandard, then consider the embeddings $\phi' : A \to \phi(I_A)B\phi(I_A)$ and $\psi' : B \to \psi(I_B)C\psi(I_B)$ are either standard or isomorphisms. Now then $\psi'|_{\phi(I_A)B\phi_A} : \phi(I_A)B\phi(I_A) \to \psi\circ\phi(I_A)\psi(I_B)C\psi(I_B)\psi\circ\phi(I_A)$ is standard by Proposition 17. Noting $\psi(I_B) \le \psi\circ\phi(I_A)$, and using the standard case, we then see $\psi'|_{\phi(I_A)B\phi_A} \circ \phi' : A \to \psi\circ\phi(I_A)C\psi\circ\phi(I_A)$ is standard (unless both are isomorphisms, in which case it is an isomorphism), and thus $\psi\circ\phi$ is substandard.

2. First assume all ϕ_n are standard. In this case, we use part 1 to show that each ρ_n is standard, and that we can write each A_n as $vN(R \cup \{p_s X_x p_s\}_{s\in\cup_{i=1}^n S_i})$ with the S_i disjoint. Then we see the inductive limit is generated by $\{R \cup \{p_s X_x p_s\}_{s\in\cup_{i=1}^\infty S_i}\}$. Then this is the interpolated free group factor with free dimension $1+\sum_{t\in\cup_{i=1}^\infty S_i} \tau(p_s)^2$, and noting we have partial sums $1+\sum_{t\in\cup_{i=1}^n S_i} \tau(p_s)^2$

of r_n, thus the inductive limit is $\lim_{n\to\infty} r_n$.

If the ϕ_n are only substandard, then consider the embeddings

$$\phi'_n : \rho_{n-1}(I_{A_1})A_n\rho_{n-1}(I_{A_1}) \to \rho_n(I_{A_1})A_{n+1}\rho_n(I_{A_1}).$$

These are standard (by the definition of substandard and Proposition 17). Thus, using the standard case, the inductive limit of $(\rho_{n-1}(I_{A_1})A_n\rho_{n-1}(I_{A_1}), \phi'_n)$ is generated by $\{R \cup \{p_s X_x p_s\}_{s\in\cup_{i=1}^{\infty} S_i}\}$. Furthermore, by part 3 of Theorem 12, each A_n is generated by $\{R_n \cup \{p_s X_x p_s\}_{s\in\cup_{i=1}^{n} S_i}\}$ where $\rho_{n-1}(I_{A_1})R_n\rho_{n-1}(I_{A_1}) = R$ and $\tau_n(p_s) = \tau_1(p_s) * \tau_n(\rho_{n-1}(I_{A_1}))$.

Thus the inductive limit (A, τ) of (A_n, ϕ_n) is generated by $\{R' \cup \{p_s X_x p_s\}_{s\in\cup_{i=1}^{\infty} S_i}\}$, where $I_{A_n} R I_{A_n} = R_n$ (noting by assumption $\tau(I_{A_n}) > 0$). Let $t_n = \tau_n(\rho_{n-1}(I_{A_1}))$ and Let $t = \lim_{n\to\infty} t_n = \tau(I_{A_1})$. Then (A, τ) is the interpolated free group factor of free dimension $1 + \sum_{s\in\cup_{i=1}^{\infty} S_i} \tau(p_s)^2 = 1 + \sum_{s\in\cup_{i=1}^{\infty} S_i} (t\tau_1(p_s))^2$. Since $r_n = 1 + \sum_{s\in\cup_{i=1}^{n} S_i} (t_i\tau_1(p_s))^2$, the free dimension of (A, τ) is $\lim_{n\to\infty} r_n$.

□

Example 22. The following example shows that the condition that $\lim_{n\to\infty} \tau_n(\rho_{n-1}(I_1))$ is needed. Let $A_n = L(F_2)$ if n is odd and $A_n = L(F_5)$ for n even, and $p \in L(F_2)$ a projection of trace $\frac{1}{2}$. Define ϕ_n for n odd to be the natural standard embedding of $L(F_2) \to L(F_3)$. For n even define ϕ_n to be the isomorphism between $L(F_5)$ and $pL(F_2)p$. This is then substandard. However, since this alternates between $L(F_2)$ and $L(F_5)$ the inductive limit cannot work as described.

Definition 23. Let $\mathcal{M}$ and $\mathcal{N}$ be two finite von Neumann algebras, with inclusion $\phi : \mathcal{N} \to \mathcal{M}$. We say that ϕ is *substandard* if for every interpolated free group factor summand F of $\mathcal{N}$, $\phi(F)$ is contained in some interpolated free group factor summand F' of $\mathcal{M}$, and the inclusion of F into F' by ϕ is substandard.

CHAPTER III

PREVIOUS RESULTS

A. Standard Free Products

The free product of two dimensional algebras, shown in Theorem 1.1 of [Dyk93], will continue to come up as a special case in many of our results. It is as follows:

Theorem 24. *Let A and B be two dimensional abelian algebras. Let p and q be the larger of the two minimal projections in A and B respectively, and let α and β be their traces. Without loss of generality, assume that $\alpha \geq \beta \geq \frac{1}{2}$. Then*

$$A * B \cong \underset{\alpha+\beta-1}{\mathbb{C}} \oplus \left(L^\infty\left(\left[0, \frac{\pi}{2}\right]\right) \otimes M_2(\mathbb{C})\right) \oplus \underset{\alpha-\beta}{\mathbb{C}}.$$

Furthermore

$$p = 1 \oplus \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus 1,$$

$$q = 1 \oplus \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \oplus 0,$$

where we represent $L^\infty([0, \pi/2]) \otimes M_2(\mathbb{C})$ as $M_2(L^\infty([0, \pi/2]))$.

Note the free dimension of A is $1 - \alpha^2 - (1-\alpha)^2 = 2\alpha - 2\alpha^2$ and similarly the free dimension of B is $2\beta - 2\beta^2$. Looking at the above theorem, we see $\operatorname{fdim}(A * B) = 1 - (\alpha + \beta - 1)^2 - (\alpha - \beta^2) = 2\alpha + 2\beta - 2\alpha^2 - 2\beta^2$, thus in this case $\operatorname{fdim}(A * B) = \operatorname{fdim}(A) + \operatorname{fdim}(B) = \operatorname{fdim}(A) + \operatorname{fdim}(B) - \operatorname{fdim}(\mathbb{C})$.

Combining Theorem 2.3 and Proposition 2.4 of [Dyk93], gives us the following:

Theorem 25. *Let $A = L(F_t) \oplus \bigoplus_{i=1}^{n} \underset{t_i}{\overset{p_i}{\mathbb{C}}}$ and $B = L(F_s) \oplus \bigoplus_{j=1}^{m} \underset{s_j}{\overset{q_j}{\mathbb{C}}}$. Assume* $\dim(A)$

and $\dim(B)$ *are at least 2. Then*

$$A * B = F \oplus \bigoplus_{y \in Y} \underset{\gamma_y}{\overset{r_y}{\mathbb{C}}},$$

where F *is either an interpolated free group factor or a diffuse hyperfinite algebra (which only occurs if* $\dim(A) = \dim(B) = 2$*),* $Y = \{(i,j) | t_i + s_j > 1\}$*, For* $(i,j) = y \in Y$*,* $\gamma_y = t_i + s_j - 1$ *and* $r_y = p_i \wedge q_i$*. Furthermore fdim*$(A * B) =$ *fdim*$(A) +$ *fdim*(B)*, which determines* F*.*

Another useful result is Proposition 3.5 from [Dyk93].

Proposition 26. *Let* $A = \overset{p_0}{L(F_t)} \oplus \bigoplus_{i=1}^{k} \underset{t_i}{\overset{p_i}{M_{n_i}}}$*. Then for* $n \geq 2$*,* $A * M_n = \overset{q_0}{L(F_s)} \oplus \bigoplus_{y \in Y} \underset{\gamma_y}{\overset{q_y}{M_n}}$*, where* $Y = \{i | \frac{t_i}{n_i} + \frac{1}{n^2} > 1\}$ *(note* Y *can have at most one element). If* Y *does have an element* i_0 *then* $\gamma = nt_{i_0} + \frac{1}{n} - n$ *and* $q_i \leq p_i$*. Furthermore fdim*$(A * M_n) =$ *fdim*$(A) + 1 - n^{-2}$*.*

Using this we prove the following result, which is Theorem 3.6 in [Dyk93], and includes the general finite dimensional case.

Theorem 27. *Let* $A = \overset{p_0}{L(F_s)} \oplus \underset{\alpha_1}{\overset{p_1}{M_{n_1}}} \oplus \cdots \oplus \underset{\alpha_k}{\overset{p_k}{M_{n_k}}}$ *and* $B = \overset{q_0}{L(F_t)} \oplus \underset{\beta_1}{\overset{q_1}{M_{m_1}}} \oplus \cdots \oplus \underset{\beta_\ell}{\overset{q_\ell}{M_{m_\ell}}}$*. Assume that neither* A *nor* B *is* $\mathbb{C}$*. Then*

$$\mathcal{M} = A * B = \overset{r}{F} \oplus \bigoplus_{y \in Y} \underset{\gamma_y}{\overset{r_y}{M_{n_y}}},$$

where F *is either an interpolated free group factor or a diffuse hyperfinite algebra (and it is only hyperfinite when* $\dim(A) = \dim(B) = 2$*), and* $Y = \{(i,j) | \alpha_i/n_i + \beta_j/n_j > 1\}$*. For* $y = (i,j) \in Y$*,* $n_y = n_i m_j$*,* $\gamma_y = m_j \alpha_i + n_i \beta_j - n_i m_j$*, and* $r_y = p_i \wedge q_j$*.*

Proof. We proceed by induction on the number of matrix factor summands with dimension greater than 1. For the base case, apply Theorem 25. For the inductive step, without loss of generality we assume that $n_1 \geq 2$. Then consider

$$A' = \overset{p_0}{L(F_s)} \oplus \underset{n_1\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \underset{\alpha_2}{M_{n_2}} \oplus \cdots \oplus \underset{\alpha_k}{M_{n_k}}.$$

If $k > 1$ or $p_0 \neq 0$ (i.e. $A' \neq \mathbb{C}$) we apply the induction hypothesis to see

$$\mathcal{M}' = A' * B = \overset{r'}{F} \oplus \bigoplus_{y \in Y'} \underset{\gamma'_y}{\overset{r'_y}{M_{n'_y}}},$$

with the values defined above. Then by Lemma 6, $p_1\mathcal{M}p_1 = p_1\mathcal{M}'p_1 * M_{n_1}$. If $k = 1$ and $p_0 = 0$ then this is true directly (with $p_1 = I_A$).

In either case we can apply Proposition 26. If $k = 1$ and $p_0 = 0$ then this completes the proof. Otherwise, there is a matrix algebra in $p_1\mathcal{M}p_1$ if and only if there exists a $y \in Y'$ of the form $(1, j)$, with $\frac{\gamma'_y}{n'_y} + \frac{\alpha_1}{n_1} > \alpha_1 n_1$, which implies $n'_y = 1$, and thus $m_j = 1$. Since $\gamma'_y = \alpha_1 n_1 + \beta_j - 1$, $\alpha_1 n_1 + \beta_j + \frac{\alpha_1}{n_1} - 1 > \alpha_1 n_1$, which is thus when $\beta_j + \frac{\alpha_1}{n_1} > 1$. Thus the matrix algebra is in $p_1\mathcal{M}p_1$ exactly when desired. Also note then that the trace of its minimal projections in $p_1\mathcal{M}p_1$ is $\frac{n_1\gamma'_y}{\alpha_1 n_1} + \frac{1}{n_1} - n_i$, and thus the trace in $\mathcal{M}$ is

$$n_1^2\alpha_1 + n_1\beta_j - n_1 + \alpha_1 - \alpha_1 n_1^2 = n_1\beta_j - n_1 + \alpha_1,$$

as desired.

Let p be the central support of p_1 in $\mathcal{M}$ (which is the same as in $A' * B$). By the properties of the free dimension contribution,

$$\mathrm{fdimC}_{\tau(p)}(p\mathcal{M}p) = \mathrm{fdimC}_{\alpha_1 n_1}(p_1\mathcal{M}p_1),$$

and by Proposition 26,

$$= \mathrm{fdimC}_{\alpha_1 n_1}(p_1\mathcal{M}'p_1) + \mathrm{fdimC}_{\alpha_1 n_1}(M_{n_1}) - \mathrm{fdimC}_{\alpha_1 n_1}(\mathbb{C})$$

$$= \mathrm{fdimC}_{\tau(p)}(p\mathcal{M}'p) - \alpha_1^2 + \alpha_1^2 n_1^2.$$

Then since $(1-p)\mathcal{M}(1-p) = (1-p)\mathcal{M}'(1-p)$ and $\mathrm{fdimC}_1(\mathcal{M}) = \mathrm{fdimC}_{1-\tau(p)}(1-p)\mathcal{M}(1-p) + \mathrm{fdimC}_{\tau(p)}p\mathcal{M}p$,

$$\mathrm{fdimC}_1(\mathcal{M}) = \mathrm{fdimC}_1(\mathcal{M}') - \alpha_1^2 + \alpha_1^2 n_1^2.$$

By the induction hypothesis this equals,

$$\mathrm{fdimC}_1(A') + \mathrm{fdimC}_1(B) - \alpha_1^2 + \alpha_1^2 n_1^2 + 1.$$

Then note $\mathrm{fdimC}_1(A) - \mathrm{fdimC}_1(A') = -\alpha_1^2 + \alpha_1^2 n_1^2$, and thus $\mathrm{fdimC}_1(\mathcal{M}) = \mathrm{fdimC}_1(A) + \mathrm{fdimC}_1 + 1$, meaning $\mathrm{fdim}(A * B) = \mathrm{fdim}(A) + \mathrm{fdim}(B)$.

□

This allows us to extend Proposition 18 in the following way (as proved in [Dyk93]).

Proposition 28. *The inclusion $L(F_r) \to L(F_r) * A$, $1 < r \leq \infty$, for A finite dimensional, is standard.*

Proof. By Theorem 27 there exists an n large enough so that $A * M_n$ is an interpolated free group factor. Take a projection $e_{11} \in L(F_r)$ with trace $\frac{1}{n}$. From this we construct a subalgebra $M_n \subseteq L(F_r)$ (with e_{11} corresponding to the standard basis element in this matrix algebra) so that $(e_{11}L(F_r)e_{11}) \otimes M_n = L(F_r)$.

Then we consider the inclusion of $e_{11}L(F_r)e_{11} \to e_{11}(L(F_r) * A)e_{11}$. By Lemma 6, $e_{11}(L(F_r) * A)e_{11} = e_{11}L(F_r)e_{11} * (e_{11}(M_n * A)e_{11})$. Since $(e_{11}(M_n * A)e_{11})$ is an interpolated free group factor, by Proposition 18 this inclusion is standard. Thus by Proposition 17 so is $L(F_r) \to L(F_r) * A$. □

Next we prove Theorem 4.6 from [Dyk93].

Theorem 29. *Let A and B be hyperfinite von Neumann algebras of dimension at least two, written as $A = \overset{p_0}{A_d} \oplus \bigoplus_{i\in I} \underset{t_i}{\overset{p_i}{M_{n_i}}}$ and $B = \overset{q_0}{B_d} \oplus \bigoplus_{j\in J} \underset{s_j}{\overset{q_j}{M_{n_j}}}$, where A_d and B_d are diffuse hyperfinite von Neumann algebras. Then*

$$A * B = F \oplus \bigoplus_{y\in Y} \underset{\gamma_y}{\overset{r_y}{M_{n_y}}},$$

where F is either an interpolated free group factor or a hyperfinite von Neumann algebra (the latter occurs only when $\dim(A) = \dim(B) = 2$), with $Y = \{(i,j)|i \in I, j \in J, t_i/n_i + s_j/n_j > 1\}$. For $(i,j) = y \in Y$, $n_y = n_i n_j$, $\gamma_y = n_j t_i + n_i t_j - n_i n_j$, and $r_y \leq p_i$, $r_y \leq q_j$. Furthermore F is determined by the formula fdim$(A * B) =$ fdim$(A) +$ fdim(B) (unless it is hyperfinite, in which case we use Theorem 24).

Proof. Let $T = \sup_{i\in I}(t_i)$ and $S = \sup_{j\in J}(s_j)$, and note these are both strictly less than one. Then let $I' = \{i \in I | t_i > 1 - S\}$ and $J' = \{j \in J | s_j > 1 - T\}$, and note these are both finite sets.

Then we can find a chain of finite dimensional subalgebras of A, $A(k)$ with the following conditions:

1. $A(k)$ is of the form $A_0(k) \oplus \bigoplus_{i\in I'} \underset{t_i}{\overset{p_i}{M_{n_i}}}$.
2. No minimal projection in $A_0(k)$ has trace greater than $1 - S$.
3. The union $\cup_{k=1}^{\infty} A(k)$ is dense in A.
4. For any k we can write $A_0(k) = M_n \oplus C$ and $A_0(k+1) = (M_n \otimes D) \oplus C$ for some finite dimensional C and D, and the inclusion of $A_0(k) \to A_0(k+1)$ is induced by the inclusion of $M_n \to M_n \otimes D$.

We construct $B(\ell)$ likewise. Then, by Theorem 27, $A(k) * B(\ell) = F_{k,\ell} \oplus \bigoplus_{y\in Y(k,\ell)} M_{n_y}$, with $F_{k,\ell}$ an interpolated free group factor. Furthermore by our choice of $A_0(k)$ and $B_0(\ell)$, $Y(k,\ell) \subseteq I' \times J'$. Thus it does not depend on k, ℓ and is exactly

Y as given in the statement of this theorem.

By Lemma 6, the embeddings of $F_{k,\ell} \to F_{k+1,\ell}$ are standard. Then by Proposition 21 the inductive limit of the $A(k) * B(\ell)$ is $L(F_s) \oplus \bigoplus_{y \in Y} \underset{\gamma_y}{\overset{r_y}{M_{n_y}}}$, where $s = \lim_{k,\ell}(\mathrm{fdim}(F_{k,\ell}))$. Since from Theorem 27, $\mathrm{fdim}(A(k)*B(\ell)) = \mathrm{fdim}(A(k)) + \mathrm{fdim}(B(\ell))$, and the matrix algebra part stays the same, $\mathrm{fdim}(A * B) = \mathrm{fdim}(A) + \mathrm{fdim}(B)$. $\square$

We end this section with Theorem 3.2 from [Dyk95].

Theorem 30. *Let* $A = \overset{p_0^A}{H^A} \oplus \bigoplus_{i \in I_A} \overset{p_i^A}{F_i^A} \oplus \bigoplus_{j \in J_A} \underset{t_j^A}{\overset{p_j^A}{M_{n_j^A}}}$ *where* H *is a diffuse hyperfinite algebra and the* F_i *are interpolated free group factors. Similarly let* $B = \overset{p_0^B}{H^B} \oplus \bigoplus_{i \in I_B} \overset{p_i^B}{F_i^B} \oplus \bigoplus_{j \in J_B} \underset{t_j^B}{\overset{p_j^B}{M_{n_j^B}}}$*. Then* $A * B = \overset{q_0}{F} \bigoplus_{y \in Y} \underset{s_y}{\overset{q_y}{M_{n_y}}}$*, where* $Y = \{(i,j), i \in J_A, j \in J_B, \frac{t_i^A}{n_i^A} + \frac{t_j^B}{n_j^B} > 1$*. For* $y = (i,j) \in Y$*,* $s_y = n_i^A t_j^B + n_j^B t_i^A - n_i^A n_j^B$ *and* $n_y = n_i^A n_j^B$*, and* F *is an interpolated free group factor (or diffuse hyperfinite algebra if* $\dim(A) = \dim(B) = 2$*). Furthermore fdim*$(A * B) =$ *fdim*$(A) +$ *fdim*(B)*, and the embeddings* $F_i^A \to F$ *are substandard.*

Proof. Enumerate the sets I_A and I_B, and let $I_{A,k}$ and $I_{B,\ell}$ be the sets of the first k and ℓ elements of I_A and I_B respectively.

Then let

$$A_k = \overset{p_0^A}{H^A} \oplus \bigoplus_{i \in I_{A,k}} \overset{p_i^A}{F_i^A} \oplus \bigoplus_{i \in I_A \setminus I_{A,k}} \overset{p_i^A}{\mathbb{C}} \oplus \bigoplus_{j \in J_A} \underset{t_j^A}{\overset{p_j^A}{M_{n_j^A}}},$$

$$B_\ell = \overset{p_0^B}{H^B} \oplus \bigoplus_{i \in I_{B,\ell}} \overset{p_i^B}{F_i^B} \oplus \bigoplus_{i \in I_B \setminus I_{B,\ell}} \overset{p_i^B}{\mathbb{C}} \oplus \bigoplus_{j \in J_B} \underset{t_j^B}{\overset{p_j^B}{M_{n_j^B}}}.$$

Let $\mathcal{M}(k,\ell) = A_k * B_\ell$. We shall prove by induction that

$$\mathcal{M}(k,\ell) = F_{k,\ell} \oplus \bigoplus_{y \in Y(k,\ell)} \underset{s_y^{k,\ell}}{\overset{q_y^{k,\ell}}{M_{n_y^{k,\ell}}}},$$

where $Y(k,\ell) = \{(x,z) | x \in J_A \cup I_A \setminus I_{A,k}, z \in J_B \cup I_B \setminus I_{B,k}, t_x^A/n_x^A + t_z^B/n_z^B > 1\}$, where $t_x^A = \tau(p_i^A)$ and $n_x^A = 1$ for $x \in I_A$ and likewise for $z \in I_B$. For $y = (x,z) \in Y(k,\ell)$, $n_y^{k,\ell} = n_x^A n_z^B$, $s_y^{k,\ell} = n_x^A t_z^B + n_z^B t_x^A - n_x^A n_z^B$, and $q_y^{k,\ell}$ is under the projections corresponding to x and y. Furthermore $\mathrm{fdim}(\mathcal{M}(k,\ell)) = \mathrm{fdim}(A_k) + \mathrm{fdim}(B_\ell)$, and the inclusion of $F_{k,\ell} \to F_{k+1,\ell}$ is substandard.

The base case, $\mathcal{M}(0,0)$, is given Theorem 29. Assuming it is true for $\mathcal{M}(k,\ell)$, consider $\mathcal{M}(k+1,\ell)$. Let i' be in $I_{A,k+1}$ but not $I_{A,k}$. Then, by Lemma 6:

$$p_{i'}^A \mathcal{M}(k+1,\ell) p_{i'}^A = p_{i'}^A \mathcal{M}(k,\ell) p_{i'}^A * F_{i'}^A.$$

Theorem 27, shows this is an interpolate free group factor. Then note that the sum of the traces of the matrix factors under $p_{i'}^A$ in $\mathcal{M}(k,\ell)$ is

$$\sum_{z,(i',z)\in Y(k,\ell)} \left(n_z^B t_z^B + (n_z^B)^2 t_{i'} - (n_z^B)^2\right) \leq 1 + t_{i'} \left(\sum_{z,(i',z)\in Y(k,\ell)} (n_z^B)^2 \right) - \left(\sum_{z,(i',z)\in Y(k,\ell)} (n_z^B)^2 \right)$$

$$= t_{i'} + (t_{i'} - 1) \left(\left(\sum_{z,(i',z)\in Y(k,\ell)} (n_z^B)^2 \right) - 1 \right) \leq t_{i'}.$$

The second inequality holds strictly, unless there is only one such z and $n_z^B = 1$. In that case the first inequality holds strictly, unless $t_z^B = 1$ In that case $B = \mathbb{C}$ which is not allowed.

This implies that $p_{i'}^A$ is not orthogonal to the interpolated free group factor in $\mathcal{M}(k,\ell)$. Since, also by Lemma 6, the central support of $p_{i'}^A$ is the same in $\mathcal{M}(k+1,\ell)$ and $\mathcal{M}(k,\ell)$, exactly those factor summands under $p_{i'}^A$ in $\mathcal{M}(k,\ell)$ are contained in this interpolated free group factor summand in $\mathcal{M}(k+1,\ell)$. This means that exactly the interpolated free group factor and the matrix algebras corresponding to i' are absorbed, as desired. Also, by Lemma 6 the embedding is substandard, as is the embedding of $F_{i'}$.

Let p be the central support of $p_{i'}^A$ in $\mathcal{M}(k,\ell)$ and $\mathcal{M}(k+1,\ell)$. Then note

$$\mathrm{fdimC}_{\tau(p)}p\mathcal{M}(r+1,\ell)p = \mathrm{fdimC}_{t_{i'}}p_{i'}^A\mathcal{M}(r+1,\ell)p_{i'}^A$$

$$= \mathrm{fdimC}_{t_{i'}}p_{i'}^A\mathcal{M}(r,\ell)p_{i'}^A + \mathrm{fdimC}_{t_{i'}}(F_{i'}^A).$$

Noting that $\mathrm{fdimC}_{t_{i'}}(p_{i'}^A A_{k+1} p_{i'}^A) - \mathrm{fdimC}_{t_{i'}}(p_{i'}^A A_k p_{i'}^A) = \mathrm{fdimC}_{t_i}(F_{i'}^A)$, this implies that $\mathrm{fdim}(\mathcal{M}(k+1,\ell)) = \mathrm{fdim}(A_{k+1}) + \mathrm{fdim}(B_\ell)$.

Since this is true for all r,ℓ and the embeddings are substandard, Proposition 21 shows it is true for the inductive limit, $\mathcal{M}$.

□

B. Amalgamated Free Products

We begin this section with a very useful tool proved by Dykema as Lemma 4.3 in [Dyk95].

Lemma 31. *Let A and B be von Neumann algebras with subalgebra D, and let $\mathcal{M} = A *_D B$. Let p be a central projection in A, and let $\underline{A} = pD \oplus (1-p)A$ and $\underline{\mathcal{M}} = \underline{A} *_D B$. Then the central support of p is the same in $\mathcal{M}$ and $\underline{\mathcal{M}}$. Furthermore $p\mathcal{M}p = vN(p\underline{\mathcal{M}}p \cup pA) = p\underline{\mathcal{M}}p *_{pD} pA$.*

Proof. Let p' be the central support of p in $\underline{\mathcal{M}}$. This means p' commutes with B, D and $\underline{A}$. Since $p \le p'$, p' also commutes with pA, thus p' commutes with A, and thus p' commutes with $\mathcal{M}$. Thus it is also the central support in $\mathcal{M}$.

Now choose partial isometries $v_i \in \underline{\mathcal{M}}$ so that $v_i^* v_i \le p$ and $\sum_{i\in I} v_i v_i^* = p' - p$, and let $p_i = v_i v_i^*$. Then $p'Ap'$ is generated by $\cup_{q,q'\in\{p,p_i\}_{i\in I}} qAq'$, and the same for B. Since p' is central in $\mathcal{M}$, $p'\mathcal{M}p'$ is generated by $p'Ap'$ and $p'Bp'$. Then $p\mathcal{M}p$ is generated by

$$\left(\bigcup_{v,v'\in\{p,v_i\}_{i\in I}} v^*Av'\right)\cup\left(\bigcup_{v,v'\in\{p,v_i\}_{i\in I}} v^*Bv'\right)$$
$$= (pA)\cup\left(\bigcup_{v,v'\in\{v_i\}_{i\in I}} v^*Av'\right)\cup\left(\bigcup_{v,v'\in\{p,v_i\}_{i\in I}} v^*Bv'\right).$$

Since $\bigcup_{v,v'\in\{v_i\}_{i\in I}} v^*Av' \subseteq \underline{A}$ we see $p\mathcal{M}p = vN(p\underline{\mathcal{M}}p \cup pA)$ (since the reverse inclusion is trivial).

Thus we need only show that that pA and $p\underline{\mathcal{M}}p$ are free with amalgamation over pD. Let $x \in p\underline{\mathcal{M}}p$ with $E_D(x) = 0$, then x is the SOT limit of a bounded sequences in the span of $p\Lambda(\underline{A}^{00}, B^{00})p$. Note any non-zero element of $\underline{A} \subseteq p\underline{\mathcal{M}}p$ is an element of pD, and thus has non-zero expectation. Thus x can be written as the SOT limit of a bounded sequences in the span of $p\left(\Lambda(\underline{A}^{00}, B^{00})\backslash\underline{A}\right)p$.

Note:

$$p\underline{A}(pA)^{00}p\underline{A} = (pD)(pA)^{00}(pD) = (pA)^{00},$$

and both $(pA)^{00}$ and $\underline{A}^{00}$ are contained in A^{00}. This lets us regroup to show that

$$\Lambda\left(p\left(\Lambda(\underline{A}^{00}, B^{00})\backslash\underline{A}\right)p, (pA)^{00}\right) \subseteq \Lambda(A^{00}, B^{00}),$$

and thus all elements have expectation zero, completing the proof.

□

In order to prove Lemma 34, which will be very useful later, we need to use the following two lemmas from [Dyk95], the first is Lemma 3.8 in that paper:

Lemma 32. *Let $\mathcal{M}$ be a von Neumann algebra, with subalgebra $\mathcal{N}$. Let $\mathcal{N} = \overset{p_0}{H} \oplus \bigoplus_{i\in I} \overset{p_i}{F_i} \oplus \bigoplus_{j\in J} \underset{t_j}{\overset{q_j}{M_{n_j}}}$, where H is a diffuse hyperfinite algebra and the F_j are interpolated free group factors. Suppose there exists p a projection in $\mathcal{N}$ and $x \in \mathcal{M}$,* *so that $x^*x = xx^* = p$, $\mathcal{M}$ is generated by $\mathcal{N}$ and x, and x is a Haar unitary, free from $p\mathcal{N}p$ in $p\mathcal{M}p$. Let p' be the central support of p in $\mathcal{N}$. Then*

$$\mathcal{M} = \overset{p'}{F} \oplus \overset{p_0p'}{H'} \oplus \bigoplus_{i\in I'} \overset{p_i}{F_i} \oplus \bigoplus_{j\in J'} \overset{q_j}{M_{n_j}},$$

where J', is the set of $j \in J$ so that $q_jp = 0$ and I' is the set $i \in I$ so that $p_ip = 0$, $F_0' = p'Hp'$, and F is an interpolated free group factor or $L(\mathbb{Z}) \otimes M_N$ (the latter case occurs only if p is minimal in $\mathcal{N}$). Furthermore, F can be determined by the fact that fdim$(\mathcal{M})$ = fdim$(\mathcal{N}) + \tau(p)^2$. For every $j \in J\backslash J'$ the inclusion into F is substandard (noting if F is not an interpolate free group factor $J = J'$), and for $j \in J'$ the inclusion into F_j is the identity (and thus substandard).

The second was proved as Lemma 4.1 in [Dyk95].

Lemma 33. *Let $\mathcal{N}$ be a von Neumann algebra with subalgebra $D = \underset{1/2}{\overset{q}{\mathbb{C}}} \oplus \underset{1/2}{\overset{q'}{\mathbb{C}}}$. Let $\mathcal{M} = \mathcal{N} *_D M_2$, where D is included as the diagonal of M_2. Suppose we have a partial isometry $w \in \mathcal{N}$ so that $p' = w^*w \le q'$ and $p = ww^* \le q$, and a von Neumann algebra $\underline{\mathcal{N}} \subseteq \mathcal{N}$ so that $p, p' \in \underline{\mathcal{N}}$ and $\mathcal{N}$ is generated by $\underline{\mathcal{N}}$ and w, $p\underline{\mathcal{N}}p' = \{0\}$, and p is minimal in $\underline{\mathcal{N}}$. Let $\underline{\mathcal{M}} = \underline{\mathcal{N}} *_D M_2$. Then there exists a partial isometry $x \in \underline{\mathcal{M}}$ such that $xx^* = p$ and $x^*x = p'$, xw^* is a Haar unitary, and it is free from $p\underline{\mathcal{M}}p$.*

We are now ready to prove Lemma 4.2 from [Dyk95].

Lemma 34. *Let $\mathcal{N} = \overset{r_0}{H} \oplus \bigoplus_{i\in I} \overset{r_i}{F_i} \oplus \bigoplus_{j\in J} \underset{t_j}{\overset{q_j}{M_{n_j}}}$, where H is a diffuse hyperfinite algebra. Let $p \in \mathcal{N}$ be a projection such that $\tau(p) = \frac{1}{2}$ and so that neither p nor $1-p$ is minimal and central in $\mathcal{N}$. Let $D = \underset{1/2}{\overset{p}{\mathbb{C}}} \oplus \underset{1/2}{\overset{1-p}{\mathbb{C}}}$. Define a matrix algebra $M_2(\mathbb{C})$, where $e_{11} = p$ and $e_{22} = 1-p$, and let $v = e_{12}$. Then*

$$\mathcal{M} = \mathcal{N} *_D M_2(\mathbb{C}) = F \oplus_{k\in K} \underset{t_k'}{\overset{q_k'}{M_{m_k}}},$$

where F is either an interpolated free group factor or a diffuse hyperfinite algebra, and

$$K = \left\{ (j,j')|j,j' \in J, j \leq p, j' \leq 1-p, \frac{t_j}{n_j} + \frac{t_{j'}}{n_{j'}} > \frac{1}{2} \right\}.$$

For $k = (j,j') \in K$, $n_k = 2n_jn_{j'}$ and $t'_k = \frac{n'_k}{2}\left(\frac{t_j}{n_j} + \frac{t_{j'}}{n_j} - \frac{1}{2}\right)$. We also know fdim$(\mathcal{M}) =$ fdim$(\mathcal{N}) + \frac{1}{4}$, which then determines F if it is an interpolated free group factor (F is only hyperfinite if $\mathcal{N}$ is dimension 4). Furthermore, the inclusion of $L(F_{s_i}) = r_i\mathcal{N}r_i$ into F is substandard. for all $i \in I$ (noting if I is not empty, then F is an interpolated free group factor).

Proof. We will do this proof in cases

Case 1: At least one of p or $1-p$ is minimal in $\mathcal{N}$.

Since we have assumed p and $1-p$ are both not minimal and central, neither of them is central (since one being central implies the other is). Thus there exists a partial isometry in $\mathcal{N}$ connecting p and $1-p$, so they are both minimal, and in fact $\mathcal{N} = M_2$, thus $\mathcal{M} = M_2 *_D M_2 = L(\mathbb{Z}) \otimes M_2$.

Let f be the centre valued trace of r_0p, and note this is in some space $L^\infty(X,\mu)$, let $r_{0,p}$ be the projection in H whose centre-valued trace is the characteristic function of $f^{-1}([1/2,1])$.

Case 2: For every $j \in J$, either $q_j \leq p$ or $q_j \leq 1-p$, for all $i \in I$, $\tau(pr_i) \leq \tau((1-p)r_i)$, and $r_{0,p} = 0$.

Then since for every $i \in I$, $\tau(pr_i) \leq \tau((1-p)r_i)$ there exists a partial isometry w_i so that $w_iw_i^* = pr_i$ and $w_i^*w_i \leq (1-p)r_i$. Likewise since the centre valued trace of pr_0 is less than $1/2$ everywhere, there is a partial isometry w_0 so that $w_0w_0^* = pr_0$ and $w_0^*w_0 \leq (1-p)r_0$. Let $w = \sum_{i \in I \cup \{0\}} w_i$. Let $\underline{\mathcal{N}} = (1-ww^*)\mathcal{N}(1-ww^*) + \mathbb{C}ww^*$. Then $\mathcal{N}$ is generated by $\underline{\mathcal{N}}$ and w.

Then, by Lemma 33, there exists have a Haar unitary u (in $ww^*\mathcal{M}ww^*$) so that $\mathcal{M}$ is generated by $\underline{\mathcal{M}} = \underline{\mathcal{N}} *_D M_2$ and u, and $ww^*\underline{\mathcal{M}}ww^*$ and u are free.

Next note that p commutes with q_j for every $j \in J$ (by assumptions of this case), and the rest of $(1 - ww^*)$ is under $1-p$, p commutes with $\underline{\mathcal{N}}$, as does $1-p$. Thus $p\underline{\mathcal{M}}p$ is generated by $p\underline{\mathcal{N}}p$ and $v\underline{\mathcal{N}}v^*$. Note $\Lambda((p\underline{\mathcal{N}}p)^0, (v\underline{\mathcal{N}}v^*)^0)$ can be rearranged to form elements of $\Lambda((\underline{\mathcal{N}})^{00}, (M_2)^{00})$, and thus have trace zero. So $p\underline{\mathcal{N}}p$ and $v\underline{\mathcal{N}}v^*$ are free.

Note

$$p\underline{\mathcal{N}}p = \overset{ww^*}{\mathbb{C}} \oplus \bigoplus_{\substack{j \in J \\ q_j \leq p}} M_{n_j},$$

$$v\underline{\mathcal{N}}v^* \cong (1-p)\underline{\mathcal{N}}(1-p) = (1-p)\mathcal{N}(1-p).$$

Then $\mathrm{fdimC}_{\frac{1}{2}} p\underline{\mathcal{N}}p = -\tau(ww^*)^2 - \sum_{\substack{j \in J \\ q_j \leq p}} t_j^2$. Since the central support of $1-p$ in $\mathcal{N}$ is $1 - \sum_{\substack{j \in J \\ q_j \leq p}} q_j^2$, we see $\mathrm{fdimC}_{\frac{1}{2}} v\underline{\mathcal{N}}v^* = \mathrm{fdimC}_1(\mathcal{N}) + \sum_{\substack{j \in J \\ q_j \leq p}} t_j^2$.

Then $p\underline{\mathcal{M}}p = p\underline{\mathcal{N}}p * v\underline{\mathcal{N}}v^*$, and thus $\mathrm{fdimC}_{\frac{1}{2}} p\underline{\mathcal{M}}p = \frac{1}{4} + \mathrm{fdimC}_1(\mathcal{N}) - \tau(ww^*)^2$. If $ww^* \neq p$ (i.e. $p\mathcal{N}p \neq \mathbb{C}$), then, by Theorem 30,

$$p\underline{\mathcal{M}}p = F_{p\underline{\mathcal{M}}p} \oplus \bigoplus_{k \in K} M_{m_k/2} \oplus \bigoplus_{j \in J'} M_{m_k},$$

where K is as given in the statement of the theorem (as is m_k for $k \in K$), and $J' = \{j \in J, p_j \leq (1-p), t_j/n_j + \tau(ww^*) > \frac{1}{2}\}$. Note the central support of ww^* in $p\underline{\mathcal{M}}p$ covers $F_{p\underline{\mathcal{M}}p} \oplus \bigoplus_{j \in J'} M_{m_k}$ but not $\bigoplus_{k \in K} M_{m_k/2}$. Furthermore the embedding of $vF_iv^* \to F_{p\underline{\mathcal{M}}p}$ is substandard.

If $p = ww^*$ then $p\underline{\mathcal{N}}p = \mathbb{C}$, however this implies that all matrix algebras are orthogonal to p. Since $\tau(pr_i) \leq \frac{1}{2}\tau(pr_i)$ for all $i \in I \cup \{0\}$, and since $\tau(p) = \frac{1}{2}$, this implies $\tau(pr_i) = \frac{1}{2}$ for all $i \in I$ and $J = \emptyset$. The free product here is trivial and we are left with

$$p\underline{\mathcal{M}}p = vHv^* \oplus \bigoplus_{i\in I} vF_iv^*.$$

In this case the central support of ww^* in $p\underline{\mathcal{M}}p$ is p. Also note that in neither case is ww^* minimal in $p\underline{\mathcal{M}}p$

If $ww^* \neq 0$, by Lemma 32:

$$p\mathcal{M}p = vN(\{u\} \cup p\underline{\mathcal{M}}p) = F_{p\mathcal{M}p} \oplus \bigoplus_{k\in K} M_{m_k/2},$$

where

$$\mathrm{fdimC}_{\frac{1}{2}}(p\mathcal{M}p) = \mathrm{fdimC}_{\frac{1}{2}}(p\underline{\mathcal{M}}p) + \tau(ww^*) = \frac{1}{4} + \mathrm{fdimC}_1(\mathcal{N}).$$

Furthermore, the inclusion of $F_{p\underline{\mathcal{M}}p} \to F_{p\mathcal{M}p}$ into is substandard if $ww^* \neq p$, and the inclusions $vF_iv^* \to F_{p\mathcal{M}p}$ are substandard if $ww^* = p$.

If $ww^* = 0$ then $\underline{\mathcal{M}} = \mathcal{M}$ and we see directly that

$$p\mathcal{M}p = p\underline{\mathcal{M}}p = F_{p\mathcal{M}p} \oplus \bigoplus_{k\in K} M_{m_k/2},$$

and

$$\mathrm{fdimC}_{\frac{1}{2}}(p\mathcal{M}p) = \mathrm{fdimC}_{\frac{1}{2}}(p\underline{\mathcal{M}}p) = \frac{1}{4} + \mathrm{fdimC}_1(\mathcal{N}).$$

In this case however, it is possible that F is hyperfinite, however this only occurs if $p\mathcal{N}p$ and $(1-p)\mathcal{N}(1-p)$ are both 2 dimensional.

Then note $\mathcal{M}$ is generated by $p\mathcal{M}p$ and v, so

$$\mathcal{M} = F \oplus \bigoplus_{k\in K} M_{m_k}.$$

and $\mathrm{fdimC}_1\mathcal{M} = \mathrm{fdimC}_{\frac{1}{2}}p\mathcal{M}p = \frac{1}{4} + \mathrm{fdimC}_1(\mathcal{N})$. so $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(\mathcal{N}) + \frac{1}{4}$. Since the inclusion of $vF_iv^* \to pFp$ is substandard, so is the inclusion of $F_i \to F$. Thus this case is complete.

Case 3: For every $j \in J$, either $q_j \leq p$ or $q_j \leq 1-p$, for all $i \in I$, and not case 1 or 2.

Let $q' = (1-p)\left(r_{0,p} + \sum_{\substack{i \in I \\ \tau(p_i p) > \frac{1}{2}\tau(p_i)}} p_i\right)$. Since we are not in case 2 (and we assume switching p with $(1-p)$ does not put us in case 2), we know $q' \neq 0$ and $q' \neq (1-p)$. Also the central support of $1-q'$ in $\mathcal{N}$ is 1. There exists a partial isometry $w \in \mathcal{N}$ so that $ww^* = q'$ and $q = w^*w < p$. Let $\underline{\mathcal{N}} = (1-q')\mathcal{N}(1-q) \oplus \overset{q}{\mathbb{C}}$. Then $\mathcal{N}$ is generated by $\underline{\mathcal{N}}$ and w.

Then, since the central support of q' in $\mathcal{N}$ is 1, we see $\mathrm{fdimC}_1(\underline{\mathcal{N}}) = \mathrm{fdimC}_1(\mathcal{N}) - \tau(q)^2$. Let $\underline{\mathcal{M}}$ be the algebra generated by v and $\underline{\mathcal{N}}$. Then we apply Lemma 33 to find a u so that $\mathcal{M}$ is generated by u and $\underline{\mathcal{M}}$, with u a Haar unitary free from $q\underline{\mathcal{M}}q$ in $q\mathcal{M}q$.

Note that $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ fall under case 2, and so

$$\underline{\mathcal{M}} = F_{\underline{\mathcal{M}}} \oplus \bigoplus_{k \in K} M_{m_k} \oplus \bigoplus_{y \in Y} M_{n_y},$$

where $Y = \{j \in J, t_j/n_j + \tau(q) > \frac{1}{2}\}$, and $\mathrm{fdim}(\underline{\mathcal{M}}) = \mathrm{fdim}(\mathcal{N}) - \tau(q)^2 + \frac{1}{4}$. Furthermore, the inclusions of the $(1-q)F_i(1-q) \to F_{\underline{\mathcal{M}}}$ are substandard. Since $\mathcal{M} = vN(\underline{\mathcal{M}} \cup \{u\})$, by Lemma 32

$$\mathcal{M} = F \oplus \bigoplus_{k \in K} M_{m_k},$$

and $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(\underline{\mathcal{M}}) = \mathrm{fdim}(\mathcal{N}) - \tau(q)^2 + \frac{1}{4}$ as desired, and the inclusion of $F_{\underline{\mathcal{M}}} \to F$ is substandard, thus the inclusion of each $F_i \to F$ is substandard, and the case is complete.

Case 4 For every $j \in J$ either $\tau(pq_j) \leq \frac{1}{2}\tau(q_j)$ or $q_j \leq p$.

Let $q = \sum_{\substack{j \in J \\ q_j \not\leq p}} pq_j$. Then there is a partial isometry $w \in \mathcal{N}$ so that $ww^* = q$ and

$q' = w^*w \leq (1-p)$. Let $\underline{\mathcal{N}} = (1-q)\mathcal{N}(1-q) \oplus \overset{q}{\mathbb{C}}$, and let $\underline{\mathcal{M}}$ be the algebra generated by $\underline{\mathcal{N}}$ and v. Once again the central support of $1-q$ in $\mathcal{N}$ is 1, so $\mathrm{fdimC}_1(\underline{\mathcal{N}}) = \mathrm{fdimC}_1(\mathcal{N}) - \tau(q)^2$.

As before we apply Lemma 33 to find a u so that u and $\underline{\mathcal{M}}$ generate $\mathcal{M}$, and u is a Haar unitary free from $q\underline{\mathcal{M}}q$ in $q\mathcal{M}q$. Also note $\underline{\mathcal{M}}$ falls under case 3, and thus:

$$\underline{\mathcal{M}} = F_{\underline{\mathcal{M}}} \oplus \bigoplus_{k\in K} M_{m_k} \oplus \bigoplus_{j\in J'} M_{n_j} \oplus \bigoplus_{y\in Y} M_{n_y},$$

where

$$J' = \{j \in J, q_j(1-p) \neq 0, t_j/\lambda_{j,q} + \tau(q) > \frac{1}{2}\},$$

$$Y = \{(j,j') \in J \times J, q_j(1-p) \neq 0, q_{j'} \leq p, t_j/\lambda_{j,q} + t_{j'}/n_{j'} > \frac{1}{2} \geq t_j/n_j + t_{j'}/n_{j'}\},$$

where $\lambda_{j,q}$ is so that $(1-q)M_{n_j}(1-q) = M_{\lambda_{j,q}}$. As before $\mathrm{fdim}(\underline{\mathcal{M}}) = \mathrm{fdim}(\mathcal{N}) + \frac{1}{4} - \tau(q)^2$ (since once again, the central support of $1-q$ in $\mathcal{N}$ is 1). Next note that Y must in fact be empty. This is because if $\lambda_{j,q} \neq n_j$ then $q_jp \neq 0$, thus M_{n_j} contains a projection of trace t_j under p and orthogonal to $q_{j'}$, thus $t_j + t_{j'} \leq \frac{1}{2}$.

Furthermore the inclusions of $F_i \to F$ are all substandard. Then, by Lemma 32,

$$\mathcal{M} = F \oplus \bigoplus_{k\in K} M_{m_k},$$

with $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(\mathcal{N}) + \frac{1}{4}$, and the inclusion of $F_{\underline{\mathcal{M}}} \to F$ is substandard, completing this case.

Case 5 All other cases.

Let $q' = \sum_{\substack{j\in J \\ \frac{1}{2}\tau(q_j)<\tau(q_jp)<\tau(q_j)}} (1-p)q_j$. There exists a partial isometry w such that $ww^* = q'$ and $q = w^*w \leq p$. Let $\underline{\mathcal{N}} = (1-q')\mathcal{N}(1-q') \oplus \overset{q'}{\mathbb{C}}$, and let $\underline{\mathcal{M}}$ be the algebra generated by $\underline{\mathcal{N}}$ and v. Then apply case 4 to $\underline{\mathcal{M}}$, to get

$$\underline{\mathcal{M}} = F_{\underline{\mathcal{M}}} \oplus \bigoplus_{k\in K} M_{m_k} \oplus \bigoplus_{j\in J'} M_{n_j} \oplus \bigoplus_{y\in Y} M_{n_y},$$

where

$$J' = \{j \in J, \tau(pq_j) > \frac{1}{2}\tau(q_j), t_j/\lambda_{j,q} + \tau(q') > \frac{1}{2}\},$$

$$Y = \{(j,j') \in J \times J, \tau(pq_j) > \frac{1}{2}\tau(q_j), q_{j'} \leq (1-p), t_j/\lambda_{j,q} + t_{j'}/n_{j'} > \frac{1}{2}\},$$

where $\lambda_{j,q}$ is so that $(1-q')M_{n_j}(1-q') = M_{\lambda_{j,q'}}$, but by the same argument as in the previous case, $Y = \emptyset$. Furthermore as before, since the central support of $1-q'$ in $\mathcal{N}$ is 1, $\mathrm{fdim}(\underline{\mathcal{M}}) = \mathrm{fdim}(\mathcal{N}) - \tau(q')^2 + \frac{1}{4}$.

Then apply Lemma 33 to find a u so that $\underline{\mathcal{M}}$ and u generate $\mathcal{M}$, and u is a Haar unitary free from $q\underline{\mathcal{M}}q$ in $q\mathcal{M}q$. Then, by Lemma 32,

$$\mathcal{M} = F \oplus \bigoplus_{k\in K} M_{m_k},$$

and $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(\mathcal{N}) + \frac{1}{4}$, and for each $i \in I$, the inclusion of $F_i \to F$ is substandard.

□

Definition 35. Let A be a multimatrix algebra $A = \oplus_{i\in I} \overset{p_i}{M}_{n_i}$ with multimatrix subalgebra $D = \oplus_{j\in J} \overset{q_j}{M}_{n_j}$. The *graph* of A with respect to D, G_D^A, is a bipartite graph with vertices I and J, and edges connecting vertices $i \in I$ and $j \in J$ if and only if $p_i q_j \neq 0$. For multimatrix algebras, A and B, with common subalgebra D, we write $G_D^{A,B}$ to denote the union of the two graphs (note this is the same as $G_D^{A\oplus B}$).

The following lemma, based on Lemma 5.2 in [Dyk95] will be used for almost all of our results on amalgamated free products.

Lemma 36. *Let $\mathcal{R}$ be a class of von Neumann algebras such that if $A \in \mathcal{R}$ and*

$p \in A$ then $pAp \in \mathcal{R}$. Let A and B be von Neumann algebras in $\mathcal{R}$ with multimatrix subalgebra D. If for all A' and B' in $\mathcal{R}$ with abelian multimatrix subalgebra D', $A' *_{D'} B'$ is of the form $H \oplus \bigoplus_{i \in I} F_i \oplus \bigoplus_{j \in J} M_{n_j}$, where H is a diffuse hyperfinite, and for $i \in I$, the F_i are interpolated free group factors, then so is $A *_D B$. Furthermore conditions on the number of I and J are preserved, and if we know fdim$(A' *_{D'} B') =$ fdim$(A') +$ fdim$(B') -$ fdim(D') then the same holds for A, B, and D.

Proof. Let $D = \bigoplus_{k \in I_D} \overset{p_k^D}{M_{n_k^D}}$. Let $\{e_{i,j}^{(k)}\}_{1 \leq i,j \leq n_k^D}$ be the standard basis for $M_{n_k^D}$. Let $e = \sum_{k \in I_D} e_{1,1}^{(k)}$. Note this is a projection in D with central support I, and eDe is abelian. Let $A' = eAe, B' = eBe$, and $D' = eDe$. Let $\mathcal{M} = A *_D B$ and let $\mathcal{M}' = vN(A' \cup B')$, and since $A' \subseteq A$, $B' \subseteq B$ we see that A' and B' are free with amalgamation over D', and thus $\mathcal{M}' = A' *_{D'} B'$. Let $V = \{e_{1,j}^{(k)} | k \in I_D, 1 < j \leq n_k^D\}$. Note then that $A = vN(A' \cup V)$, $B = vN(B' \cup V)$, and $D = vN(D' \cup V)$. Then $\mathcal{M} = vN(\mathcal{M}' \cup V)$. Thus $\mathcal{M}' = e\mathcal{M}e$. By our assumption we know A' and B' are in $\mathcal{R}$ and D' is an abelian multimatrix algebra. Thus by assumption $\mathcal{M}'$ is of the correct form, which implies $\mathcal{M}$ is of the same form with the same number of factor summands of each type. Now if $\mathrm{fdim}(A' *_{D'} B') = \mathrm{fdim}(A') + \mathrm{fdim}(B') - \mathrm{fdim}(D')$, then $\mathrm{fdimC}_{\tau(e)}(A' *_{D'} B') = \mathrm{fdimC}_{\tau(e)}(A') + \mathrm{fdimC}_{\tau(e)}(B') - \mathrm{fdimC}_{\tau(e)}(D')$. Then since the central support of e is the identity $\mathrm{fdimC}_1(A *_D B) = \mathrm{fdimC}_1(A) + \mathrm{fdimC}_1(B) - \mathrm{fdimC}_1(D)$ and thus $\mathrm{fdim}(A *_D B) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$.

□

The amalgamated free product of multimatrix algebra was described by Dykema as Theorem 5.1 in [Dyk95].

Theorem 37. *Let A and B be multimatrix algebras with subalgebra D. Let $\mathcal{M} = A *_D B$. Then $\mathcal{M} = H \oplus \bigoplus_{i \in I} F_i \oplus \bigoplus_{j \in J} M_{n_i}$ where H is a diffuse hyperfinite algebra and the F_i are interpolated free group factors. Furthermore fdim$(\mathcal{M}) =$ fdim$(A) +$*

fdim$(B) -$ *fdim*(D). *If* D *is finite dimensional then* J *is finite and* H *is type I. If* A *and* B *are also finite dimensional then* I *is also finite.*

Proof. First, by Lemma 36 without loss of generality assume $D = \bigoplus_{k\in I_D} \underset{t_k}{\overset{p_k^D}{\mathbb{C}}}$.

Let $A = \bigoplus_{i\in I_A} \underset{t_i}{\overset{p_i}{M_{n_i}}}$ and $B = \bigoplus_{i\in I_B} \underset{t_i}{\overset{p_i}{M_{n_i}}}$, for disjoint sets I_A and I_B. Without loss of generality assume D embeds diagonally in A and B.

Then define $\mathcal{N}(k) = vN(p_k^D A p_k^D \cup p_k^D B p_k^D) = (p_k^D A p_k^D) * p_k^D B p_k^D$. By Theorem 29, we see $\mathcal{N}(k) = F_k \oplus \bigoplus_{i\in I_k} M_{n_i}$ where F_k is either an interpolated free group factor or a diffuse type I hyperfinite algebra, and $I_k = \{(i,j)|i \in I_A, j \in I_B, t_i/\lambda_{i,k} + t_j/\lambda_{j,k} > t_k^D, p_k^D p_i \neq 0, p_k^D p_j\}$ where $p_k^D M_{n_i} p_k^D = M_{\lambda_{i,k}}$. If p_k^D is not minimal in either of A or B then I_k is finite and $F_k \neq \{0\}$.

Let $\mathcal{N} = \bigoplus_{k\in I_D} \mathcal{N}(k)$. Thus

$$\mathcal{N} = \bigoplus_{k\in I_D} F_k \oplus \bigoplus_{k\in I_D, i\in I_k} M_{n_i}.$$

Choose an order for I_D. Define a set

$$S = \left\{v_{k,i} | i \in I_A \cup I_B, k \in I_D, p_k^D p_i \neq 0, \exists k' < k, p_{k'}^D p_i \neq 0\right\},$$

where each $v_{k,i}$ is a partial isometry in M_{n_i} so that $v_{k,i} v_{k,i}^* \leq p_k^D$ and $v_{k,i}^* v_{k,i} \leq p_{k'}^D$ for some $k' < k$, and $v_{k.i} v_{k,i}^*$ is minimal in M_{n_i}. Then $\mathcal{M} = vN(\mathcal{N} \cup S)$. For $S' \subseteq S$ define $\mathcal{N}(S') = vN(\mathcal{N} \cup S')$. Put an order on S and let S_j be the first j elements of S. Our goal is to show that each $\mathcal{N}(S_j)$ is of the desired format, and that the the inclusion of $\mathcal{N}(S_j)$ into $\mathcal{N}(S_{j+1})$ is substandard. Furthermore we shall show that $\mathrm{fdim}(\mathcal{N}(S_j)) = \mathrm{fdim}(\mathcal{N}(S_{j-1})) + \tau(v_j v_j^*)^2$, where v_j is the jth element of S.

We proceed by induction. The base case, $j = 0$, $S_j = \emptyset$, $\mathcal{N}(S_0) = \mathcal{N}$ is clear from the definition of $\mathcal{N}$. Now assume it is true for $\mathcal{N}(S_{j-1})$, and let

$$\mathcal{N}(S_{j-1}) = \overset{p_0^{j-1}}{H_{j-1}} \oplus \bigoplus_{i \in I_{j-1}} \overset{p_i^{j-1}}{F_{i,j-1}} \oplus \bigoplus_{y \in Y_{j-1}} \underset{t_y^{j-1}}{\overset{p_y^{j-1}}{M_{n_y}}}.$$

Let $v_j = v_{k,i}$ be the jth element of S. Let $\bar{v_j} = v_j v_j^* + v_j^* v_j$. Note the central support of $\bar{v_j}$ is the same in $\mathcal{N}(S_j)$ and $\mathcal{N}(S_{j-1})$, so we examine $\bar{v_j}\mathcal{N}(S_j)\bar{v_j}$ and $\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}$. Clearly $\bar{v_j}\mathcal{N}(S_j)\bar{v_j}$ is generated by $\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}$ and v_j. We claim that the algebra generated by v_j, $\overset{\bar{v_j}}{M_2}$, is free with amalgamation over $C = \overset{v_j v_j^*}{\mathbb{C}} \oplus \overset{v_j^* v_j}{\mathbb{C}}$ from $\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}$ in $\bar{v_j}\mathcal{N}(S_j)\bar{v_j}$.

Without loss of generality assume $v_j \in A$. To avoid confusion we will use A^{0C} to denote elements of A with expectation onto C of zero, A^{0D} to denote those with expectation onto D of zero. Note any element $\mathcal{N}(S_{j-1})$ is the SOT limit of a bounded sequence in the span of $\Lambda(A_{j-1}^{0D}, B_{j-1}^{0D})$ and D where A_{j-1} is generated by $\bigoplus_{k \in I_D} p_k^D A p_k^D$ and $A \cap S_{j-1}$, and B_{j-1} is defined likewise. Note that $\bar{v_j} A_{j-1} \bar{v_j} = C$, thus $(\bar{v_j} A_{j-1} \bar{v_j})^{0C} = \{0\}$. Thus

$$\Lambda\left((\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j})^{0C}, M_2^{0C} \right) = \Lambda\left((\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}) \setminus A_{j-1})^{0C}, M_2^{0C} \right).$$

Also note $A_{j-1} M_2^{0C} A_{j-1} \subseteq A_j^{0D}$, thus rearranging we see that

$$\Lambda\left((\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}) \setminus A_{j-1})^{0C}, M_2^{0C} \right) \subseteq \bar{v_j} \Lambda\left(A_j^{0D}, B_{j-1}^{0D} \right) \bar{v_j}.$$

By freeness with amalgamation this has expectation onto D of zero, and thus has trace zero (by Lemma 4). Thus M_2 and $\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}$ are free with amalgamation over C.

Thus as long as neither $v_j v_j^*$ nor $v_j^* v_j$ are minimal and central in $\bar{v_j}\mathcal{N}(S_{j-1})\bar{v_j}$ we can apply Lemma 34. If one of them is minimal and central, without loss of generality assume $v_j v_j^*$, then $\bar{v_j}\mathcal{N}(S_j)\bar{v_j} = \left(v_j^* v_j \mathcal{N}(S_{j-1}) v_j^* v_j \right) \otimes M_2(\mathbb{C})$. Note this means that

$v_j v_j^* \leq p_{y'}^{j-1}$ for some $y' \in Y_{j-1}$. Then

$$\mathcal{N}(S_j) = \overset{p_0^j}{H_j} \oplus \bigoplus_{i \in I_j} \overset{p_i^j}{F_{i,j}} \oplus \bigoplus_{y \in Y_j} \overset{p_y^j}{M_{n_y}},$$

where $Y_j = Y_{j-1} \backslash \{y'\}$ and $I_j = I_{j-1}$. Those p_i^{j-1}, p_{y-1}^j, and p_0^j not orthogonal to $v_j^* v_j$ are expanded appropriately. Note then each $F_{i,j-1}$ is embedded into $F_{i,j}$ and the embedding is just a dilation so it is substandard.

In this case, since $v_j^* v_j \mathcal{N}(S_j) v_j^* v_j = v_j^* v_j \mathcal{N}(S_{j-1}) v_j^* v_j$, and the central support of $v_j^* v_j$ in $\mathcal{N}(S_j)$ is $\bar{v}_j$,

$$\text{fdimC}_{\tau(\bar{v}_j)} \bar{v}_j \mathcal{N}(S_j) \bar{v}_j = \text{fdimC}_{\tau(v_j^* v_j)} v_j^* v_j \mathcal{N}(S_{j-1}) v_j^* v_j$$

$$= \text{fdimC}_{\tau(\bar{v}_j)} \bar{v}_j \mathcal{N}(S_{j-1}) \bar{v}_j - \text{fdimC}_{\tau(v_j v_j^*)} v_j v_j^* \mathcal{N}(S_{j-1}) v_j v_j^*$$

$$= \text{fdimC}_{\tau(\bar{v}_j)} \bar{v}_j \mathcal{N}(S_{j-1}) \bar{v}_j + \tau(v_j v_j^*)^2.$$

Since outside of the central support of $\bar{v}_j$ there is no change this tells us that $\text{fdim}(\mathcal{N}(S_j)) = \text{fdim}(\mathcal{N}(S_{j-1})) + \tau(v_j v_j^*)^2$.

If neither are minimal and central, then Lemma 34 tells us that

$$\bar{v}_j \mathcal{N}(S_j) \bar{v}_j = F \bigoplus_{y \in Y} M_{n_y},$$

where Y corresponds to pairs of $y_1, y_2 \in Y_{j-1}$ so that $p_{y_1}^{j-1}$ is orthogonal to $v_j v_{j^*}$ and not $v_j^* v_j$ and vice-versa for $p_{y_2}^{j-1}$, where $t_{y_1}^{j-1}/\lambda_{y_1,\bar{v}_j} + t_{y_2}^{j-1}/\lambda_{y_2,\bar{v}_j} > \tau(v_k v_k^*)$, where $\bar{v}_j M_{n_{y_1}^{j-1}} \bar{v}_j =<_{\lambda_{y_1,\bar{v}}}$ (and likewise for $\lambda_{y_2,\bar{v}_j}$).

The properties of this algebra can also be determined by Lemma 34, but we will not go into the details here. Thus,

$$\mathcal{N}(S_j) = \overset{p_0^j}{H_j} \oplus \bigoplus_{i \in I_j} \overset{p_i^j}{F_{i,j}} \oplus \bigoplus_{y \in Y_j} \overset{p_y^j}{M_{n_y}},$$

where Y_j is formed by taking all the $y \in Y_{j-1}$ so that p_y^{j-1} is orthogonal to $\bar{v}_j$, and keeping the corresponding matrix algebras as they were and also all the $y \in Y$ and expanding those matrix algebras as necessary. I_j will correspond to all the $i \in I_{j-1}$ so that p_i^{j-1} is orthogonal to $\bar{v}_j$, plus one new one which combines all those that were not orthogonal to $\bar{v}_j$, as well as possibly some of the matrix algebras and a diffuse hyperfinite part.

Furthermore Lemma 34 tells us that the embedding of each of the interpolated free group factors $F_{i,j-1}$ not orthogonal to $\bar{v}_j$ into the new interpolated free group factor is substandard.

Lemma 34 also tells us that

$$\mathrm{fdim}\mathrm{C}_{\tau(\bar{v}_j)}\bar{v}_j\mathcal{M}(S_j)\bar{v}_j = \mathrm{fdim}\mathrm{C}_{\tau(\bar{v}_j)}\bar{v}_j\mathcal{M}(S_{j-1})\bar{v}_j + \tau(\bar{v}_j)^2/4$$

$$= \mathrm{fdim}\mathrm{C}_{\tau(\bar{v}_j)}\bar{v}_j\mathcal{M}(S_{j-1})\bar{v}_j + \tau(v_j v_j^*)^2,$$

which completes the proof of the claim.

Thus the inductive limit is as desired, since all the necessary embeddings are standard and the free dimension adds up correctly at each stage.

Since the number of matrix algebras only decreases, if A and B (and consequently D) were finite dimensional, it is clear that the number of matrix algebras will remain finite.

So to complete the proof we must show that if D is finite dimensional then there are a finite number of interpolated free group factor summands. To start with, there are a finite number of interpolated free group factor summands in $\mathcal{N}$, since in each $\mathcal{N}(k)$ there is at most one interpolated free group factor summand. Furthermore the number of interpolated free group factor summands can only increase when we add a $v \in S$ so that $\bar{v}$ is orthogonal to all existing interpolated free group factors and neither vv^* nor v^*v are minimal and central in $\bar{v}\mathcal{N}(S_v)\bar{v}$ (where S_v is a the subset of S of elements which are added before v).

First we claim that if p_k^D is not minimal in A or B, then for any v with $vv^* \leq p_k^D$, vv^* and F_k are not orthogonal. We can do this by carefully examining the traces, which is done in the proof of Proposition 47 and so not included here. Let $S_1 \subseteq S$ be the set of $v \in S$ so that $v^*v \leq p_k^D$ and $vv^* \leq p_{k'}^D$ where both p_k^D and $p_{k'}^D$ are minimal in at least one of A and B. Let $S_2 \subseteq S_1$ be those for which neither vv^* nor v^*v are in D. Then $\mathcal{N}(S_2)$ has the same number of interpolated free group factors as $\mathcal{N}$ (it is easy to check that for $v \in S_2$ either vv^* or v^*v will be minimal and central in $\bar{v}\mathcal{N}(S_2 \backslash \{v\})\bar{v}$). Furthermore $S_1 \backslash S_2$ is finite, since for any $k \in I_D$ every v so that $vv^* = p_k^D$ or $v^*v = p_k^D$ must come from at most one matrix algebra in A or at most one matrix algebra in B, and so there are only finitely many per k and only finitely many k. We have already established that adding any $v \in S \backslash S_1$ does not increase the number of interpolated free group factors. Thus there is a finite number of them.

$\square$

Example 38. The following example shows that we can have an infinite number of matrix factors in the product of two multimatrix algebras despite having connected graph and finite dimensional D. Let $A = \underset{1/3}{\overset{p_0}{M_2}} \oplus \bigoplus_{i=1}^{\infty} \underset{\frac{1}{6\cdot 2^i}}{\overset{p_i}{M_2}}$ and $B = \underset{1/6}{\overset{q_1}{M_2}} \oplus \underset{2/3}{\overset{q_2}{\mathbb{C}}}$, and $D = \underset{1/3}{\overset{q_1}{\mathbb{C}}} \oplus \underset{2/3}{\overset{q_2}{\mathbb{C}}}$, where $p_i \leq q_2$ for all $i \geq 1$, $q_1 \leq p_0$, and $\tau(p_0 q_2) = 1/3$. The free product is:

$$\underset{1/6}{M_4} \oplus \bigoplus_{i=1}^{\infty} \underset{\frac{1}{6\cdot 2^i}}{M_2}.$$

Definition 39. The set $\mathcal{R}_1$ (referred to as $\mathcal{R}$ in [Dyk11]) is the finite direct sum of the following types of von Neumann algebras:

1. Finite Dimensional algebras.
2. $L^\infty([0,1]) \otimes M_n$.
3. The hyperfinite II_1 factor.
4. Interpolated free group factors.

In [Dyk11], as Theorem 4.4, Dykema shows this is closed under amalgamated free products over finite dimensional algebras.

Theorem 40. *Let A and B be von Neumann algebras in $\mathcal{R}_1$ with finite dimensional subalgebra D. Then $A*_D B$ is in $\mathcal{R}_1$ and $fdim(A*_D B) = fdim(A)+fdim(B)-fdim(D)$.*

The proof of this is nearly identical to that of Theorem 51. This uses a weaker base case, and proceeds by induction on both the number of interpolated free group factors and hyperfinite II_1 factors (as opposed to Theorem 51, which does induction only the number of interpolated free group factors).

It is also clear from the fact that $\mathrm{fdim}(A *_D B) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$ that if A and B do not contain copies of $L(F_\infty)$ then neither does $A *_D B$ (this class is referred to as $\mathcal{R}_0$ in [Dyk11]).

CHAPTER IV

AMALGAMATION OVER A FINITE DIMENSIONAL SUBALGEBRA

A. Useful Lemmas

Lemma 41. *Let $\mathcal{M}$ be a hyperfinite von Neumann algebra with separable predual and finite dimensional abelian subalgebra D. Then there exists a chain of finite dimensional subalgebras in $\mathcal{M}$ containing D whose union is dense in $\mathcal{M}$.*

Proof. $\mathcal{M}$ can be written as the direct sum of a type I part and a type II part, each of which can be approximated separately, so we deal with these cases individually.

Case I, $\mathcal{M}$ is type I:

Let $\mathcal{M} = \bigoplus_{i\in I} M_{m_i} \otimes \overset{p_i}{L^\infty}(X_i, \mu_i)$, for finite probability measures μ_i and countable set I. Without loss of generality we may assume that all elements of D are diagonal matrices in $\bigoplus_{i\in I} M_{m_i}(L^\infty(X_i, \mu_i))$. Write $D = \bigoplus_{k\in I_D} \underset{t_k^D}{\overset{p_k^D}{\mathbb{C}}}$, where I_D is a finite index set.

In order to construct the chain of subalgebras we require, we use partitions $P_{n,i}$ of X_i. We refer to a partition $P_{n,i}$ of X_i as compatible with M_{m_i} if for every $e_{jj} \in M_{m_i}$, and $S \in P_{n,i}$ there exists a $k \in I_D$ so $e_{jj} \otimes \chi_S \leq p_k^D$. Since m_i and I_D are finite we can refine any partition of X_i to one compatible with M_{m_i}.

Set $A_0 = D$, and let $I_n \subseteq I$ consist of the first n elements of I. Set $P_{0,i} = \{X_i\}$ for all $i \in I$ and inductively choose $P_{n,i}$ to be a partition of X so that:

1. The measure, μ_i, of every set in $P_{n,i}$ is less than $1/2^n$ or is composed of a single atom.

2. The partition $P_{n,i}$ is compatible with D for M_{m_i}.

3. $P_{n,i}$ refines $P_{n-1,i}$.

Set

$$A_n = \left(\bigoplus_{i \in I_n} M_{m_i} \otimes \overset{p_i}{\ell^\infty}(P_{n,i}, \mu_i) \right) \oplus \bigoplus_{k \in I_D} \overset{p_k^D - p_k^D\left(\sum_{i \in I_n} p_i\right)}{\mathbb{C}}.$$

Then $D = A_0 \subseteq A_1 \subseteq \dots$ and $\cup A_i$ is dense in $\mathcal{M}$.

Case II, $\mathcal{M}$ is type II.

We assume $D = \overset{p_1}{\mathbb{C}} \oplus \overset{p_2}{\mathbb{C}}$, and from there the general case follows inductively. Write $\mathcal{M} = L^\infty(X, \mu) \otimes R$ where R is the hyperfinite II_1 factor and μ is a finite probability measure. Take a sequence of matrix algebras M_{2^k} spanned by the standard basis elements $e_{i,j}^{(k)}$ with the inclusion of M_{2^k} into $M_{2^{k+1}}$ by the map which takes $e_{i,j}^{(k)}$ to $e_{2i-1,2j-1}^{(k+1)} + e_{2i,2j}^{(k+1)}$, and so that $\cup_k M_{2^k}$ is dense in R.

Recall projections in finite von Neumann algebras are equivalent exactly when they have the same centre-valued trace. Let $f \in L^\infty(\mu)$ be the centre-valued trace of p_1. Note f takes values only in $[0, 1]$. We can construct a projection which has centre-valued trace f in the following way:

$$p_1' = \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} e_{(2i-1),(2i-1)}^{(k)} \otimes \chi_{f^{-1}\left(\left(\frac{2i-1}{2^k}, \frac{2i}{2^k}\right]\right)}.$$

To show this has the desired centre-valued trace, we examine the partial sums:

$$p_{1,N}' = \sum_{k=1}^{N} \sum_{n=1}^{2^{k-1}} e_{(2i-1),(2i-1)}^{(k)} \otimes \chi_{f^{-1}\left(\left(\frac{2i-1}{2^k}, \frac{2i}{2^k}\right]\right)}.$$

Note $p_{1,N}' \in L^\infty(X, \mu) \otimes M_{2^N}$. Consider elements of $L^\infty(X, \mu) \otimes M_{2^N}$ as M_{2^N} valued functions on X, and let F_N be the function corresponding to $p_{1,N}'$. Then, up to a set of measure zero, for each $x \in X$ if $f(x) = y$, then $F_N = \sum_{i < 2^N y} e_{i,i}^N$. Noting that the centre-valued trace is $Tr_{2^N}(F_N)$ (where Tr_{2^N} is the normalized trace on M_{2^N}), and thus is the simple function f_N which takes only values of the form $i/2^N$ and approximates f from below. Thus $\|f_N - f\|_\infty \leq 1/2^N$, and so p_1' has the desired centre-valued trace, and is thus equivalent to p_1. So without loss of generality, by modifying our choice of dense subalgebra, assume p_1 is of this form.

As in the type I case we will construct the chain of subalgebras using a sequence of partitions P_i of X. First set $P_1 = \{S_{1,1}, S_{1,2}\}$ where $S_{1,1} = f^{-1}([0,1/2])$ and $S_{1,2} = f^{-1}((1/2,1])$, and define $\phi_{1,n}$ to be the characteristic function of $S_{1,n}$. Next define A_1 to be the algebra generated by

$$B_0 = \{(e_{1,1}^{(1)} \otimes \phi_{1,2}), (e_{2,2}^{(1)} \otimes \phi_{1,1}), (p_1 - e_{1,1}^{(1)} \otimes \phi_{1,2}), (p_2 - e_{2,2}^{(1)} \otimes \phi_{1,1})\}.$$

Note these are four orthogonal projections, so they generate $\mathbb{C}^4$ which is clearly finite dimensional. It is also clear that p_1 and $1 - p_1$ are in this algebra, thus $D \subseteq A_1$.

Next inductively construct partitions $P_k = \{S_{k,1}, \ldots, S_{k,\ell}\}$, satisfying the following conditions:

1. For every $S_{k,n} \in P_k$ either $\mu(S_{k,n}) < 1/2^k$ or $S_{k,n}$ is composed of a single atom.
2. For each $S_{k,n} \in P_k$, $f(S_{k,n}) \subseteq (i/2^k, (i+1)/2^k]$ for some integer i.
3. P_k refines P_{k-1}.

Let $\phi_{k,n}$ be the characteristic function of $S_{k,n}$. Let $r_{k,n}$ be the integer such that $f(S_{k,n}) \subseteq (r_{k,n}/2^k, (r_{k,n}+1)/2^k]$. Then let A_k be the algebra generated by the basis:

$$\Bigg\{ e_{i,j}^{(k)} \otimes \phi_{k,n}, \left(p_1\phi_{k,n} - \sum_{m=1}^{r_{k,n}} (e_{m,m}^{(k)} \otimes \phi_{k,n}) \right), \\ \left(p_2\phi_{k,n} - \sum_{m=r_{k,n}+2}^{2^k} (e_{m,m}^{(k)} \otimes \phi_{k,n}) \right) | S_{k,n} \in P_{k,n}, i \neq r_{k,n}, j \neq r_{k,n} \Bigg\}.$$

By our representation of p_1, for $S_{k,n} \in P_k$ if $i < r_{k,n} + 1$ then $e_{i,i}^{(k)} \otimes \phi_{k,n} \leq p_1$ and if $i > r_{k,n}+1$, $e_{i,i}^{(k)} \otimes \phi_{k+1,n} \leq p_2$, thus A_k is

$$\bigoplus_{S_{k,n}\in P_k} \left(\underset{2^{-k}\mu(S_{k,n})}{\overset{\sum_{i\neq r_{k,n}} (e_{i,i}^{(k)}\otimes\phi_{n,k})}{M_{2^k-1}}} \oplus \underset{\int_{S_{k,n}} f d\mu - \frac{r_{k,n}}{2^k}\mu(S_{k,n})}{\overset{\left(p_1\phi_{k,n}-\sum_{m=1}^{r_{k,n}}(e_{m,m}^{(k)}\otimes\phi_{k,n})\right)}{\mathbb{C}}} \oplus \underset{\int_{S_{n,k}}(1-f)d\mu-\frac{2^k-r_{k,n}-1}{2^k}\mu(S_{n,k})}{\overset{\left(p_2\phi_{k,n}-\sum_{m=r_{k,n}+2}^{2^k}(e_{m,m}^{(k)}\otimes\phi_{k,n})\right)}{\mathbb{C}}} \right).$$

Noting $p_1\phi_{k,n} - \sum_{i=1}^{r_{k,n}}(e_{i,i}^{(k)} \otimes \phi_{k,n})$ is in A_k, as is each of the summands, so is $p_1\phi_{k,n}$. Since this is true for each $S_{k,n} \in P_k$, so is p_1. By the same argument so is p_2 thus $D \subseteq A_k$.

Next we check that $A_{k-1} \subseteq A_k$. Take any of the $e_{i,j}^{(k-1)} \otimes \phi_{k-1,n}$ with $i, j \neq r_{k-1,n}$, then

$$\sum_{\substack{n', \\ S_{k,n'} \subseteq S_{k-1,n}}} \left((e_{2i-1,2j-1}^{(k)} \otimes \phi_{k,n'}) + (e_{2i,2j}^{(k)} \otimes \phi_{k,n'}) \right) = (e_{i,j}^{(k-1)} \otimes \phi_{k-1,n}).$$

Since for any n', $p_1\phi_{k,n'}$ is in A_k and $p_1\phi_{k-1,n} = \sum_{n', S_{k,n'} \subseteq S_{k-1,n}} p_1\phi_{k,n'}$, $p_1\phi_{k-1} \in A_k$. Since all the summands in $p_1\phi_{k-1,n} - \sum_{i=1}^{r_{k-1,n}}(e_{i,i}^{(k-1)} \otimes \phi_{k-1,n})$ are in A_k, so is it. Thus $A_{k-1} \subseteq A_k$.

Furthermore $\cup_{k=1}^{\infty} A_k$ is dense in $\mathcal{M}$. The restriction on the measure of $S_{k,n}$ ensures the approximation of $L^\infty(\mu)$. Since R is approximated by $M_{2^{k+1}-1}$, which approximates M_{2^k} as $k \to \infty$, we see that $\mathcal{M} = \cup_{k=1}^{\infty} A_k$.

$\square$

Lemma 42. *Let $\mathcal{N} = (M_m \oplus M_{n-m} \oplus B) *_D C$ and $\mathcal{M} = (M_n \oplus B) *_D C$, where B, C are finite von Neumann algebras and $D = \bigoplus_{i=1}^{K} \overset{p_i^D}{\mathbb{C}}$ with $K \in \mathbb{N} \cup \{\infty\}$. $\mathcal{N}$ is included in $\mathcal{M}$ by including M_m and M_{n-m} as blocks on the diagonal of M_n, and B and C by the identity. Assume there exists a partial isometry in $\mathcal{N}$ between minimal projections in M_m and M_{m-n} (for example if there exists a factor $\mathcal{F}$ with $M_m \oplus M_{n-m} \subseteq \mathcal{F} \subseteq \mathcal{N}$). Then for any minimal projection $p \in M_m$ such that $p \leq p_i^D$ for some i, $p\mathcal{N}p * L(\mathbb{Z}) \cong$*

$p\mathcal{M}p$.

Proof. Denote $A = M_m \oplus M_{n-m} \oplus B$ and $A' = M_n \oplus B$. We choose a representation of the matrices so that D embeds diagonally and $p = e_{11}$.

Note $\mathcal{M} = vN(\mathcal{N} \cup \{e_{1n}\})$. From our assumptions there exists a partial isometry $v \in \mathcal{F} \subseteq \mathcal{N}$ such that $v^*v = e_{11}$ and $vv^* = e_{nn}$. Let $u = v^*e_{n1}$, and thus $uu^* = u^*u = e_{11}$.

Since u and $\mathcal{N}$ generate $\mathcal{M}$, u and $e_{11}\mathcal{N}e_{11}$ generate $e_{11}\mathcal{M}e_{11}$. To prove that u and $e_{11}\mathcal{N}e_{11}$ are $*-$free, we must prove that any element of $\Lambda(\{u^k | k \in \mathbb{Z}\backslash\{0\}\}, (e_{11}\mathcal{N}e_{11})^0)$ has zero trace.

By breaking u into v^*e_{n1}, and noting $v \in \mathcal{N}$, and $v = e_{nn}ve_{11}$ we see that

$$u(e_{11}\mathcal{N}e_{11})^0 u = v^*e_{n1}(e_{11}\mathcal{N}e_{11})^0 v^*e_{n1} \subseteq (e_{11}\mathcal{N}e_{nn})(e_{1n})(e_{11}\mathcal{N}e_{nn})(e_{n1}),$$

$$u^*(e_{11}\mathcal{N}e_{11})^0 u = e_{1n}v(e_{11}\mathcal{N}e_{11})^0 v^*e_{n1} \subseteq (e_{1n})(e_{nn}\mathcal{N}e_{nn})^0(e_{n1}),$$

$$u(e_{11}\mathcal{N}e_{11})^0 u^* = v^*e_{n1}(e_{11}\mathcal{N}e_{11})^0 e_{1n}v \subseteq (e_{11}\mathcal{N}e_{nn})(e_{1n})(e_{11}\mathcal{N}e_{11})^0(e_{1n})(e_{nn}\mathcal{N}e_{11}),$$

$$u^*(e_{11}\mathcal{N}e_{11})^0 u^* = e_{1n}v(e_{11}\mathcal{N}e_{11})^0 e_{1n}v \subseteq (e_{1n})(e_{nn}\mathcal{N}e_{11})(e_{1n})(e_{nn}\mathcal{N}e_{11}).$$

Thus rearranging tells us that

$$\begin{aligned}&\Lambda(\{u^k | k \in \mathbb{Z}\backslash\{0\}\}, (e_{11}\mathcal{N}e_{11})^0)\\ &\qquad\qquad \subseteq \Lambda(\{e_{1n}, e_{n1}\}, (e_{11}\mathcal{N}e_{11})^0 \cup e_{11}\mathcal{N}e_{nn} \cup e_{nn}\mathcal{N}e_{11} \cup (e_{nn}\mathcal{N}e_{nn})^0).\end{aligned}$$

Any $x \in \mathcal{N}^{00}$ is the SOT limit of a bounded sequence in the span of $\Lambda(A^{00}, C^{00})$. Since non-zero elements of $e_{11}Ae_{11}$ or $e_{nn}Ae_{nn}$ have non-zero trace, elements of $(e_{11}\mathcal{N}e_{11})^0$ and $(e_{nn}\mathcal{N}e_{nn})^0$ are SOT limits of bounded sequences in the span of $\Lambda(A^{00}, C^{00})\backslash A^{00}$ (and thus have expectation zero also). Similarly note that since

$e_{11}Ae_{nn} = 0 = e_{nn}Ae_{11}$, elements of $e_{11}\mathcal{N}e_{nn}$ and $e_{nn}\mathcal{N}e_{11}$ can be written in the same way.

Thus we can approximate any element in

$$\Lambda(\{e_{1n}, e_{n1}\}, \{(e_{11}\mathcal{N}e_{11})^0, e_{11}\mathcal{N}e_{nn}, e_{nn}\mathcal{N}e_{11}, (e_{nn}\mathcal{N}e_{nn})^0\})$$

by bounded sequences in the span of $\Lambda(\{e_{n1}, e_{1n}\}, \mathrm{span}(\Lambda(A^{00}, C^{00})\backslash A^{00}))$. Now note that $E_D(Ae_{n1}A) = E_D(Ae_{1n}A) = \{0\}$. Thus

$$\Lambda(\{e_{n1}, e_{1n}\}, \mathrm{span}(\Lambda(A^{00}, C^{00})\backslash A)) \subseteq \mathrm{span}(\Lambda(A'^{00}, C^{00})).$$

By the definition of freeness with amalgamation, elements of $\Lambda(A'^{00}, C^{00})$ have expectation zero, and thus trace zero, and so u and $e_{11}\mathcal{N}e_{11}$ are $*$-free and u is a Haar unitary.

Thus $e_{11}\mathcal{N}e_{11} * L(Z) \cong e_{11}\mathcal{M}e_{11}$.

$\square$

Lemma 43. *Let $\mathcal{M} = ((M_n \otimes A) \oplus B) *_D C$ and $\mathcal{N} = (M_n \oplus B) *_D C$ for $A, B,$ and C von Neumann algebras with finite trace, and $D = \bigoplus_{i=1}^{K} \overset{p_i^D}{\mathbb{C}}, k \in \mathbb{N} \cup \{\infty\}$, where C, B, M_n have expectations onto D and where $E_D^{M_n \otimes A} = E_D^{M_n} \otimes \tau_A$. Let p be a minimal projection in M_n, with $p \leq p_i^D$ for some i. Then $p\mathcal{N}p * A \cong p\mathcal{M}p$.*

Proof. Since pA and $\mathcal{N}$ generate $\mathcal{M}$, $p\mathcal{N}p$ and A (embedded as pA) generate $p\mathcal{M}p$. Thus we need only show that these two algebras are $*$-free.

As in the previous proof, traceless elements of $p\mathcal{N}p$ are expectationless, thus $(p\mathcal{N}p)^0 = (p\mathcal{N}p)^{00}$. Traceless elements of $p(M_n \oplus B)p$ are zero, thus $(p(M_n \oplus B)p)^0 = (p(M_n \oplus B)p)^{00} = \{0\}$. Elements of $(p\mathcal{N}p)^{00}$ are the SOT limits of bounded sequences in $\mathrm{span}(p\Lambda((M_n \oplus B)^{00}, C^{00})p)$ which is the same as $\mathrm{span}(p(\Lambda((M_n \oplus B)^{00}, C^{00})\backslash(M_n \oplus$

$B)^{00})p)$. Thus $\Lambda((pA)^0, (p\mathcal{N}p)^0)$, can be approximated by elements of

$$\Lambda((pA)^0, \mathrm{span}(p(\Lambda((M_n \oplus B)^{00}, C^{00})\backslash(M_n \oplus B)^{00})p)).$$

Any traceless element of $a \in pA$ multiplied on either or both of the left and the right by elements of M_n is expectationless, since the result, thought of as a matrix in $M_n(A)$, has entries which are all scalar multiples of a. This means $(M_n \oplus B)(pA)^0(M_n \oplus B) \subseteq ((M_n \otimes A) \oplus B)^{00}$. Thus elements of $\Lambda((pA)^0, \mathrm{span}(p(\Lambda((M_n \oplus B)^{00}, C^{00})\backslash(M_n \oplus B)^{00})p))$ are contained in the span of $\Lambda(((M_n \otimes A) \oplus B)^{00}, C^{00})$.

By the definition of amalgamated free product elements of $\Lambda(((M_n \otimes A) \oplus B)^{00}, C^{00})$ are expectationless and traceless, pA and $p\mathcal{M}p$ are $*-$free. $\square$

Definition 44. We call an embedding $\phi : \mathcal{M} \to \mathcal{N}$ a *simple step* if it follows one of the two following patterns:

1. $\mathcal{N} = \overset{p}{A} \oplus \overset{q}{B}$, $\mathcal{M} = \left(\bigoplus_{i=1}^n \overset{p_i}{A}\right) \oplus \overset{q}{B}$, with $p = \sum_{i=1}^n p_i$, and $\phi(a, b) = (a, \ldots, a, b)$. We call this a *simple step of the first kind.*
2. $\mathcal{N} = \underset{t}{M_n} \oplus \underset{t}{M_m} \oplus \overset{p}{B}$, $\mathcal{M} = \underset{t}{M_{n+m}} \oplus \overset{p}{B}$ where $\phi(x, y, b) = \left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, b\right)$. We call this a *simple step of the second kind.*

Lemma 45. *Let $\mathcal{N}$ and $\mathcal{M}$ be two finite dimensional von Neumann algebras. A trace preserving embedding, $\phi : \mathcal{N} \to \mathcal{M}$, can be written as a the composition of a finite sequence of simple steps.*

Proof. Since $\mathcal{N}$ and $\mathcal{M}$ are both finite dimensional we write them as direct sums of matrix algebras, $\mathcal{N} = \bigoplus_{i \in I} \underset{s_i}{\overset{p_i}{M_{n_i}}}$ and $\mathcal{M} = \bigoplus_{j \in J} \underset{t_j}{\overset{q_j}{M_{m_j}}}$, with I and J disjoint. The *Bratteli diagram* for this embedding is a graph with vertex set $J \cup I$. There is an edge between $i \in I$ and $j \in J$ if $\phi(p_i)q_j \neq 0$. The edges are decorated with the *partial* *multiplicity* of M_{n_i} in M_{m_j}, which is defined as $\sqrt{\dim(p_i M_{m_j} p_i)}/n_i$, or equivalently $\tau_{\mathcal{M}}(\phi(p_i)q_j)/(n_i t_j)$. Denote this $\lambda_{i,j}$.

A standard property of the Bratteli diagram of a unital embedding is that for any $j \in J$ $m_j = \sum_{i \in I} \lambda_{i,j} n_i$ (more generally, if the embedding is not unital $m_j \geq \sum_{i \in I} \lambda_{i,j} n_i$).

Then to decompose this map into simple steps. For each edge connecting $i \in I$ and $j \in J$, we use a simple step of the first kind to map M_{n_i} to create $\lambda_{i,j}$ copies of it, trace t_j and which lie under $p_i q_j$. Call the resulting algebra $\mathcal{N}_1$. The Bratteli diagram of the inclusion of $\mathcal{N}_1$ into $\mathcal{M}$ by ϕ has exactly one edge for each matrix factor summand in $\mathcal{N}_1$. Then use simple steps of the second kind to link all the matrix algebras in $\mathcal{N}_1$ which are connected to the same vertex $j \in J$. Then the composition of these simple steps gives us ϕ.

$\square$

Definition 46. We define the *graph* G_D^A of a hyperfinite von Neumann Algebra A and multimatrix subalgebra $D = \oplus_{i \in I_D} \overset{p_i^D}{M_{n_i}}$ to be the graph with vertex set I_D where vertices $i, j \in I_D$ are connected by an edge if $p_j A p_i \neq 0$.

We also use $G_D^{A,B}$ to denote the union of the graphs G_D^A and G_D^B, where B is another von Neumann algebra with subalgebra D.

Remark. The graph as defined for multimatrix algebras in Definition 35 is connected if and only if the graph in Definition 46 is connected, however it is not the same graph. In particular we have replaced the vertices corresponding to matrix summands of A with edges that connect any pair of vertices which both have edges to that vertex.

B. Amalgamation over Finite Dimensions

Proposition 47. *Let A and B be hyperfinite von Neumann algebras with finite trace, with finite dimensional subalgebra D. If the graph $G_D^{A,B}$ is connected, and no minimal projection in D is also minimal in A or B, then $A *_D B = F \oplus_{r=1}^{R} M_{n_r}$ where F is either an interpolated free group factor or a diffuse type I hyperfinite algebra (the latter can only occurs when D is a factor and A and B are finite dimensional). Furthermore $fdim(A *_D B) = fdim(A) + fdim(B) - fdim(D)$.*

Proof. Lemma 36 allows us to assume without loss of generality that D is abelian.

Write $A = \overset{p_d^A}{A_d} \oplus \overset{p_a^A}{A_a}$ and $B = \overset{p_d^B}{B_d} \oplus \overset{p_a^B}{B_a}$, where A_a and B_a are the type I atomic parts, and A_d and B_d the diffuse parts. Write $D = \oplus_{k \in I_D} \underset{t_k^D}{\overset{p_k^D}{\mathbb{C}}}$, $A_a = \oplus_{\ell \in I_{A_a}} \underset{t_\ell}{\overset{p_\ell}{M_{n_\ell}}}$, and $B_a = \oplus_{\ell \in I_{B_a}} \underset{t_\ell}{\overset{p_\ell}{M_{n_\ell}}}$, with I_D, I_{A_a}, and I_{B_a} disjoint.

Use Lemma 41 to construct a chain $A_d(i)$ of finite dimensional subalgebras of A_d containing $p_d^A D p_d^A$ and whose union is dense in A_d. By Lemma 45 we may assume that the inclusions of the chain are simple steps. Let $A_d(i) = \oplus_{\ell \in I_{A_d(i)}} \underset{t_\ell}{\overset{p_\ell}{M_{n_\ell}}}$, and $B_d(j) = \oplus_{\ell \in I_{B_d(i)}} \underset{t_\ell}{\overset{p_\ell}{M_{n_\ell}}}$. Note $I_D, I_{A_d(i)}$, and $I_{B_d(i)}$, are finite, but I_{A_a} and I_{B_a} may not be.

Also, since A_d is diffuse, we choose the $A_d(i)$ in such a way that the trace of the any minimal projection $p \in A_d(i)$, $p \leq p_k^D$ is less than $t_k^D - \tau(p')$ for any minimal projection $p' \in B_a$, $p' \leq p_k^D$, and also less than $\frac{t_k^D}{2}$. This is possible, since the minimal projections $p' \in B_a, p' \leq p_k^D$ have traces strictly less than t_k^D (and since they are summable, their supremum is strictly less too). We construct $B_d(j)$ in the same way. Denote $A(i) = A_d(i) \oplus A_a$ and $B(j) = B_d(j) \oplus B_a$. Note $A(i) = \oplus_{\ell \in I_{A(i)}} \underset{t_\ell}{M_{n_\ell}}$ where $I_{A(i)} = I_{A_d(i)} \cup I_{A_a}$. We also start our sequences far enough along that $G_D^{A(i),B(j)}$ is connected.

First we prove that for any i and j,

$$\mathcal{M}(i,j) := A(i) *_D B(j) = \overset{p_0^{\mathcal{M}}}{F_{i,j}} \oplus \bigoplus_{r=1}^{R} \overset{p_r^{\mathcal{M}}}{M_{n_r}},$$

where $F_{i,j}$ is either an interpolated free group factor or a diffuse type I hyperfinite algebra, and $F_{i,j}$ is the only part that depends on i and j. After that we will show that the inclusion $F_{i,j} \to F_{i+1,j}$ is standard.

In order to prove the first claim, we note that $\mathcal{M}(i,j)$ is the amalgamated free product of two multimatrix algebras, which is described in Theorem 37. As in the proof of that theorem, we construct $\mathcal{M}(i,j)$ by first defining $\mathcal{N}_{i,j}(k) = vN(p_k^D A(i) p_k^D \cup p_k^D B(j) p_k^D)$ for each $k \in I_D$. Since $vN(p_k^D A(i) p_k^D \cup p_k^D B(j) p_k^D) = p_k^D A(i) p_k^D * p_k^D B(j) p_k^D$, by Theorem 29 this is $F_k(i,j) \oplus \bigoplus_{y \in Y(k,i,j)} M_{n_y}(\mathbb{C})$, where $F_k(i,j)$ is either an interpolated free group factor or a diffuse type I hyperfinite algebra. Each element of $y \in Y(k,i,j)$ corresponds to a pair (ℓ, ℓ'), $\ell \in I_{A(i)}, \ell' \in I_{B(j)}$, representing a pair of matrix algebras in $A(i)$ and $B(j)$, so that $t_\ell/\lambda_{\ell,k} + t^B_{\ell'}/\lambda_{\ell',k} > t_k^D$, where $\lambda_{\ell,k}$ is the partial multiplicity of p_k^D in M_{n_ℓ} (i.e. $M_{\lambda_{\ell,k}} \cong p_k^D M_{n_\ell} p_k^D$). Since $\lambda_{\ell',k} \geq 1$ and $\lambda_{\ell,k} \geq 1$, $t_\ell + t^B_{\ell'} > t_k^D$. For $\ell \in I_{A_d(i)}$, $t_\ell + t_{\ell'} \leq p_k^D$, thus if $(\ell, \ell') \in Y(k,i,j)$ then $\ell \in I_{A_a}$ and $\ell' \in I_{B_a}$. Thus the matrix algebras in the $\mathcal{N}_{i,j}(k)$ are determined entirely by the atomic parts of A and B, and thus does not depend on i or j.

We use the term *connector* to refer to a partial isometry v in $A(i)$ (or $B(j)$) so that vv^* and v^*v are minimal in $A(i)$ (or $B(j)$) and so $vv^* \leq p_k^D$ and $v^*v \leq p_{k'}^D$ for some $k, k' \in I_D$.

Let $\mathcal{N}_{i,j} = \bigoplus_{k \in I_D} \mathcal{N}_{i,j}(k)$. The construction of $\mathcal{M}(i,j)$ proceeds by choosing a set of connectors, S, so that $\mathcal{M}(i,j) = \mathcal{N}_{i,j}(S) := vN(\mathcal{N}_{i,j} \cup S)$. In this construction all connector $v \in S$, are such that $vv^* \leq p_k^D$ and $v^*v \leq p_{k'}^D$ for $k \neq k'$.

We next show that for any connector v, with $vv^* \leq p_k^D$, vv^* is not orthogonal to $F_k(i,j)$. Without loss of generality, assume $v \in A(i)$. Then $vv^* \in \mathcal{N}_{i,j}(k)$, and is a minimal projection in $p_k^D A(i) p_k^D$, and thus $vv^* \leq p_{\ell_v}$ for some $\ell_v \in I_{A(i)}$. Using Theorem 29, we determine the traces of the matrix factor summands of $\mathcal{N}_{i,j}(k)$ which are not orthogonal to vv^*, i.e. those with index (ℓ_v, ℓ) for some $\ell \in I_{B_a}$. If there are any, this implies $\ell_v \in I_{A_a}$, (if $\ell \in I_D$, it is clear that vv^* is not orthogonal to $F_k(i,j)$, as it is contained in it). For (ℓ_v, ℓ) to be the index of a matrix factor summand of $\mathcal{N}_{i,j}(k)$, $t_{\ell_v}/\lambda_{\ell_v,k} + t_\ell/\lambda_{\ell,k} > t_k^D$, where $\lambda_{\ell,k}$ is the partial multiplicity of p_k^D in M_{n_ℓ}. Since $t_\ell, t_{\ell_v} < t_k^D$ this can only happen if at least one of $\lambda_{\ell_v,k}$ and $\lambda_{\ell,k}$ is one. Let L be the set of $\ell \in B_a$ so that $t_{\ell_v}/\lambda_{\ell_v,k} + t_\ell/\lambda_{\ell,k} > t_k^D$.

First assume $\lambda_{\ell_v,k} = 1$. Then for each $\ell \in L$ we have a matrix algebra $\underset{\lambda_{\ell,k}(t_{\ell_v} + \frac{t_\ell}{\lambda_{\ell,k}} - t_k^D)}{M_{\lambda_{\ell,k}}}$. Since $\lambda_{\ell,k} t_{\lambda_{\ell,k}} = \tau(p_k^D I_{M_\ell} p_k^D)$, and $p_k^D I_{M_\ell} p_k^D \leq p_k^D$, the total trace of these is

$$\sum_{\ell \in L} \lambda_{\ell,k}^2 (t_{\ell_v} + \frac{t_\ell}{\lambda_{\ell,k}} - t_k^D) \leq t_{\ell_v} \left(\sum_{\ell \in L} \lambda_{\ell,k}^2 \right) + t_k^D - t_k^D \left(\sum_{\ell \in L} \lambda_{\ell,k}^2 \right)$$

$$= t_{\ell_v} - (t_k^D - t) \left(\left(\sum_{\ell \in L} \lambda_{\ell,k}^2 \right) - 1 \right) \leq t_{\ell_v}.$$

Now since $t_k^D > t$, unless $|L| = 1$ and $\lambda_{\ell,k} = 1$, the second inequality holds strictly. If $|L| = 1$ and $\lambda_{\ell,k} = 1$, the first inequality holds strictly, since otherwise $\tau(p_k^D) = \tau(p_k^D I_{M_\ell} p_k^D)$ and this is minimal in $B(j)$, thus p_k^D is minimal in B, contradicting our assumption. Thus the total trace of the matrix factor summands of $\mathcal{N}_{i,j}(k)$ not orthogonal to vv^* is strictly less than that of vv^*, and thus $F_k(i,j)$ is not orthogonal to vv^*.

Instead assume $\lambda_{\ell_v,k} > 1$ and thus $\lambda_{\ell,k} = 1$ for each $\ell \in L$. Let $N = |L|$, then for each such $\ell \in L$ we have a matrix algebra $\underset{\lambda_{\ell_v,k}(\frac{t_{\ell_v}}{\lambda_{\ell_v,k}} + t_\ell - t_k^D)}{M_{\lambda_{\ell_v,k}}}$. Thus the total trace of these is:

$$\sum_{\ell \in L} \lambda_{\ell_v,k}^2 \left(\frac{t_{\ell_v}}{\lambda_{\ell_v,k}} + t_\ell - t_k^D \right) \leq t_{\ell_v} N \lambda_{\ell_v,k} + \lambda_{\ell_v,k}^2 t_k^D - \lambda_{\ell_v,k}^2 N t_k^D$$

$$= t_{\ell_v}\lambda_{\ell_v,k} - \lambda_{\ell_v,k}(N-1)(\lambda_{\ell_v,k}t_k - t_{\ell_v}) \leq t_{\ell_v}\lambda_{\ell_v,k}.$$

Then if $N > 1$, the second inequality holds strictly. As before, if $N = 1$, then the first inequality holds strictly, otherwise p_k^D would be minimal in B. Since vv^* is a minimal projection in $M_{\lambda_{\ell_v,k}}$ as a factor summand of $\mathcal{N}_{i,j}(k)$, we can form $p_1, \ldots, p_\lambda$, minimal projections in $\mathcal{N}_{i,j}(k)$ which are all equivalent to vv^* in $\mathcal{N}_{i,j}(k)$ and mutually orthogonal. Thus the trace of their sum is $t_{\ell_v}\lambda_{\ell_v,k}$, which we have established is strictly greater than the total trace of all the matrix factor summands not orthogonal to them, and so they must not be orthogonal to $F_k(i,j)$, and so vv^* is not either.

For a connector $v \in S$ we let $\bar{v} = vv^* + v^*v$. Put an order on the elements of S, and let S_v be the set of all those that come before v. Then our construction of $\mathcal{M}(i,j)$ proceeds by adjoining the v in order, and noting $\bar{v}\mathcal{N}_{i,j}(S_v \cup \{v\})\bar{v} = vN(\bar{v}\mathcal{N}_{i,j}(S_v)\bar{v} \cup \{v\})$. In particular note that $vN(\bar{v}\mathcal{N}_{i,j}(S_v)\bar{v} \cup \{v\}) \cong \bar{v}\mathcal{N}_{i,j}(S_v)\bar{v} \underset{\overset{vv^*}{\mathbb{C}} \oplus \overset{v^*v}{\mathbb{C}}}{*} M_2$, where M_2 is generate by v. This is given by Lemma 34 (in general we have to check if either vv^* or v^*v are minimal and central in $\bar{v}\mathcal{N}_{i,j}(S_v)\bar{v}$, but here we have shown that $F_k(i,j)$ is not orthogonal to either, so neither can be minimal).

It also shows that the matrix algebras all correspond to pairs of matrix algebras in $\mathcal{N}_{i,j}(S_v)$, and are determined by the properties of those matrix algebras and the connectors which are not orthogonal to them. We have shown that all matrix algebras in $\mathcal{N}_{i,j}$ are determined entirely by A_a and B_a. Since all connectors in $A_d(i)$ and $B_d(i)$ are contained in the interpolated free group factors in $\mathcal{N}_{i,j}$, none of them affect the matrix factors summands. Thus the matrix factor summands of $\mathcal{M}(i,j)$ are determined only by the atomic parts of A and B, and thus do not depend on i or j.

Lemma 34 also tells us that in $\bar{v}\mathcal{N}_{i,j}(S_v \cup \{v\})\bar{v}$ there is exactly one interpolated free group factor (it cannot be a hyperfinite algebra, since $\bar{v}\mathcal{N}_{i,j}(S_v)\bar{v}$ contains at least some of $F_k(i,j)$, and thus is not dimension 4). Thus for any $v \in S$, $vv^* \leq p_k^D$ and

$v^*v \leq p_{k'}^D$ for $k, k' \in I_D$, then $F_k(i,j)$ and $F_{k'}(i,j)$ are contained in some F which is an interpolated free group factor summand of $\mathcal{M}(i,j)$ (noting if F_k is hyperfinite, then $p_k^D A(i) p_k^D = \overset{p_1}{\mathbb{C}} \oplus \overset{p_k^D - p_1}{\mathbb{C}}$ and $p_k^D B(i) p_k^D = \overset{p_2}{\mathbb{C}} \oplus \overset{p_k^D - p_2}{\mathbb{C}}$, and thus $vv^* \in \{p_1, p_2, 1-p_1, 1-p_2\}$. By Theorem 24 all of these have central support which contains I_{F_k}). Then since the graph $G_D^{A(i),B(j)}$ is connected, there is exactly one free group factor summand of $\mathcal{M}(i,j)$, unless, $A = \mathbb{C} \oplus \mathbb{C}$, $B = \mathbb{C} \oplus \mathbb{C}$, and $D = \mathbb{C}$, in which case it is a type I diffuse hyperfinite algebra.

Having shown $\mathcal{M}(i,j)$ is of the correct form, we next show that the inclusion of $F_{i,j} \to F_{i+1,j}$ is standard. We know $A_d(i) \to A_d(i+1)$ is a simple step. If it is of the first kind, then it makes copies of a matrix algebra M_{n_ℓ} in $A_d(i)$ contained in the free group factor summand $F_{i,j}$ in $\mathcal{M}(i,j)$. Lemma 43 shows that $F_{i,j} \to F_{i+1,j}$ is a standard embedding.

Alternately if it is of the second kind, taking $M_{n_\ell}, M_{n_{\ell'}} \in A_d(i)$, into $M_{\ell+\ell'} \in A_d(i+1)$, since M_{n_ℓ} and $M_{n_{\ell'}}$ are contained in $F_{i,j}$, we apply Lemma 42 to show $F_{i,j} \to F_{i+1,j}$ is a standard embedding.

By Theorem 37 for any i and j, $\mathrm{fdim}(\mathcal{M}(i,j)) = \mathrm{fdim}(A(i)) + \mathrm{fdim}(B(j)) - \mathrm{fdim}(D)$. By Proposition 21 since the embeddings of $F_{i,j} \to F_{i+1,j}$ are standard, the free dimension of the inductive limit of the $\mathcal{M}_{i,j}$ is the limit of the free dimensions of the $\mathcal{M}(i,j)$. Thus the inductive limit, $A *_D B$ has free dimension of the sum equal to that of A and B minus that of D.

Note that since the matrix portion remains constant it can be computed by computing an earlier stage in the chain, using Theorem 37. Once the matrix algebras have been determined, the free dimension determines the rest. $\square$

Theorem 48. *Let A and B be hyperfinite von Neumann Algebras, with finite dimensional subalgebra D. Then $A *_D B = H \oplus \bigoplus_{i \in I} F_i$ where H is a hyperfinite* *algebra, F_i are interpolated free group factors, and I is a finite index set. Furthermore fdim$(A *_D B) =$ fdim$(A) +$ fdim$(B) -$ fdim(D).*

Proof. As usual, Lemma 36 allows us to assume D is abelian. As in the previous proposition set $A = A_a \oplus A_d$ and $B = B_a \oplus B_d$. We choose our sequence in the same way as Proposition 47, except that when we ensure the minimal projections $p \in A_d(i)$, $p \leq p_k^D$ have smaller trace than $t_k^D - \tau(p')$, for all minimal projections $p' \in B$, $p' \leq p_k^D$, we must exclude the cases where $p' = p_k^D$ (in the context of Proposition 47 these did not exist). In this case (where p_k^D is minimal in B) we only require that $\tau(p) \leq t_k^D/4$ for all $p \in A, p \leq p_k^D$.

Unfortunately it is no longer true that we must have only one free group factor summand, nor that all of $A_d(i)$ and $B_d(j)$ must be contained in interpolated free group factors. Nor is it still true that all elements of $p_k^D A_d(i) p_k^D$ for some k must be in the same interpolated free group factor.

Instead, for each free group factor summand F in $\mathcal{M}(i,j)$, we shall associate a subset $Q_F \subseteq I_D$, so that $Q_F \cap Q_{F'} = \emptyset$ if $F \neq F'$. Furthermore if F is not orthogonal to both A_d and B_d then Q_F is non-empty and if $k \in Q_F$ then $p_k^D F' p_k^D = 0$ for all $F' \neq F$.

Each $k \in I_D$ can be assigned to a Q_F in one of two ways, or not assigned at all. *Method I* is used if p_k^D is not minimal in either A or B. Then, as in Proposition 47, there is one interpolated free group factor summand F of $\mathcal{M}(i,j)$ which contains $p_k^D A_d(i) p_k^D$ and $p_k^D B_d(i) p_k^D$, we assign k to Q_F.

Next consider $k \in I_D$ where p_k^D is minimal in either A or B. Without loss of generality assume it is B and assume p_k^D is not orthogonal to A_d (noting that it is orthogonal to B_d).

As in the previous proof, and the construction from Theorem 37, we construct

$\mathcal{M}(i,j)$ from $\mathcal{N}_{i,j}(k) = vN(p_k^D A(i) p_k^D \cup p_k^D B(i) p_k^D)$, and then set $\mathcal{N}_{i,j} = \oplus_{k \in I_D} \mathcal{N}_{i,j}(k)$. Set $\mathcal{N}_{i,j}(S) = vN(\mathcal{N}_{i,j} \cup S)$ where S is a set of connectors.

Then there is a set $S \subseteq (A(i) \cup B(j))$ of connectors so that $\mathcal{M}(i,j) = \mathcal{N}_{i,j}(S)$. If there is a connector v such that either v^*v or vv^* equals p_k^D, then $v \in B_a$, since p_k^D is minimal in B and not in $A(i)$. Without loss of generality we can assume that there is at most one $v \in S$ so that vv^* or v^*v is p_k^D (without loss of generality assume vv^*), since if there is another w so that $ww^* = p_k^D$ we can replace it with v^*w.

We use *Method II* when there is such a v and v^*v is not minimal and central in $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$ where $\bar{q} = vv^* + v^*v$ and $S_0 = S \backslash \{v\}$.

For this to be consistent, we need to show that this does not depend on the choice of v. Assume there exists v and v', where $vv^* = v'v'^* = p_k^D$. Now assume v^*v is minimal and central in $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$. Take v'^*v', with S_0 as before. Then there must be some $w \in \mathcal{N}_{i,j}(S_0)$ such that $ww^* = v^*v$ and $w^*w = v'^*v'$. Note since v^*v is minimal and central in $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$, it is minimal and commutes with p_k^D in $\mathcal{N}_{i,j}(S_0)$. Thus it must be in some factor summand of $\mathcal{N}_{i,j}(S_0)$ which is a matrix algebra orthogonal to p_k^D. But using w, we see v'^*v' must be in this same matrix factor summand and has the same trace, so it must be minimal and thus central in $\bar{q}'\mathcal{N}_{i,j}(S_0)\bar{q}'$.

The construction in Theorem 37 shows us that $\bar{q}\mathcal{M}(i,j)\bar{q} = vN(\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q} \cup \{v\})$, and furthermore that this is isomorphic to $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q} \underset{\mathbb{C} \oplus \mathbb{C}}{\overset{vv^* \quad v^*v}{*}} M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is generated by v.

Since v^*v is not minimal and central, Lemma 34 tells us that $\bar{q}\mathcal{M}(i,j)\bar{q}$ is of the form $F \oplus \bigoplus_\ell M_{m_\ell}$ where F is an interpolated free group factor (by our assumption on the size of projections in $A(i)$ we know that F is not a hyperfinite algebra and non-zero). Then F is contained in some free group factor summand F' in $\mathcal{M}(i,j)$, and we assign k to $Q_{F'}$. Thus any free group factor in $\mathcal{M}(i,j)$ other than F' is orthogonal to $\bar{q}$, and thus to p_k^D.

Since we have assumed v^*v is not minimal and central in $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$, it is not minimal there. If it were minimal, but not central, then there would be some partial isometry w also in $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$, $w \in A(i)$ (we know v is the only one in B) so that $w^*w \leq v^*v$, and $ww^* \leq vv^*$, in which case, since v^*v is minimal, $w^*w = v^*v$, and thus (since the traces are the same) $ww^* = vv^*$, and so $p_k^D = ww^*$, and this is minimal in A, contradicting our condition that p_k^D is minimal only in A.

Thus there must be some value $\delta > 0$ so that $\tau(v^*v) - \tau(p) > \delta$ for all minimal projections $p \in \bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$, $p \leq v^*v$. We may start our sequence $A_d(i)$ far enough along so that any minimal projection in $A_d(i)$ has trace less than δ (noting that the choice of δ will still hold further along in the chain). Lemma 34 tells us that matrix algebras in $\bar{q}\mathcal{N}_{i,j}(S)\bar{q}$ arise from pairs of matrix algebras $\underset{t}{M_n}$ and $\underset{s}{M_m}$, under vv^* and v^*v respectively with $t/n + s/m > \tau(vv^*)$. By our assumption if M_n is not orthogonal to A_d, then $t < \delta$, which implies $t/n + s/m \leq \tau(vv^*)$. Thus $p_k^D A_d(i) p_k^D \subseteq F'$ where $k \in Q_{F'}$. Start our sequence far enough along that this is true for each $k \in I_D$ assigned with Method II.

Any $k \in I_D$ not assigned by the two above methods will remain unassigned. We next show that for any interpolated free group factor summand F of $\mathcal{M}(i,j)$, if $Q_F = \emptyset$ then F is orthogonal to $A_d(i)$ and $B_d(j)$.

To do this we show that any summand factor of $A_d(i)$ or $B_d(i)$ is either contained in an interpolated free group factor F with a non-empty set Q_F, or orthogonal to all interpolated free group factor summands in $\mathcal{M}(i,j)$. So start by taking any $\ell \in I_{A_d(i)}$ or $I_{B_d(j)}$ (without loss of generality assume $I_{A_d(i)}$). Let $K_\ell = \{k \in I_D | p_k^D M_{n_\ell} p_k^D \neq 0\}$. If there is any $k \in K_\ell$ and factor summand F in $\mathcal{M}(i,j)$ such that $k \in Q_F$, then $M_{n_\ell} \subseteq F$ (as shown when describing Method I and Method II).

If there is no such $k \in K_\ell$, i.e. $k \notin Q_F$ for all F, then we claim that M_{n_ℓ} is orthogonal to all free group factor summands of $\mathcal{M}(i,j)$.

For each $k \in K$, p_k^D is minimal in B, otherwise it would have been assigned by Method I. For each $k \in K_\ell$, where p_k^D is not central in B, choose a connector $v_k \in B_a$ such that $v_k v_k^* = p_k^D$, and let V be the set of these connectors. Now there cannot be any $v_k \in V$ such that $v_k v_k^* = p_k^D$ and $v_k^* v_k = p_{k'}^D$ for $k, k' \in K_\ell$, because choosing a connector $w \in M_{n_\ell} \subseteq A_d(i)$ so that $w^* w \leq p_k^D$ and $ww^* \leq p_{k'}^D$, we would have $v_k^* v_k$ not central in $\bar{q}\mathcal{N}_{i,j}(S_0)\bar{q}$, meaning k would have been assigned by Method II, contrary to the hypothesis.

We choose S to be a set of connectors in $\mathcal{M}(i,j)$ containing V, so that $\mathcal{N}_{i,j}(S) = \mathcal{M}(i,j)$ and such that for each $k \in K$ there is at most one $v \in S$ such that either vv^* or v^*v equals p_k^D.

Then M_{n_ℓ} is a factor summand in $\mathcal{N}_{i,j}(S \backslash V)$, since every p_k^D not orthogonal to M_{n_ℓ} is minimal in B, thus $\mathcal{N}_{i,j}(k) = p_k^D A(i) p_k^D$, and only the connectors from $A(i)$ affecting this have been added. We now order $V = \{v_{k(1)}, \ldots, v_{k(r)}\}$. We consider the chain of embeddings,

$$\mathcal{N}_{i,j}(S\backslash V) \to \mathcal{N}_{i,j}((S\backslash V) \cup \{v_{k(1)}\}) \to \cdots \to \mathcal{N}_{i,j}((S\backslash V) \cup \{v_{k(1)}, \ldots, v_{k(r)}\}) = \mathcal{M}(i,j).$$

In each of them the algebra M_{n_ℓ} is embedded as the corner of a factor summand which is also a matrix algebra. Thus M_{n_ℓ} is orthogonal to all free group factor summands in $\mathcal{M}(i,j)$.

Note that there may be a number of unassigned $k \in I_D$ This means that $p_k^D A_d(i) p_k^D$ and $p_k^D B_d(j) p_k^D$ (only one of which is non-zero) may have parts in one or more interpolated free group factor summands of $\mathcal{M}(i,j)$ with non-empty Q_F, and in matrix algebras.

If Method I or II is applied to some $k \in I_D$, in $\mathcal{M}(i,j)$, it will also be applied in $\mathcal{M}(i+1,j)$. Furthermore if there is some factor summand F in $\mathcal{M}(i,j)$ so that

$k, k' \in Q_F$, then there must be a partial isometry $v \in \mathcal{M}(i,j)$ so that $vv^* \leq p_k^D$ and $v^*v \leq p_{k'}^D$, and both vv^* and v^*v are in either $A_d(i)$ or $B_d(j)$ (though v may not be). Then, since v is embedded in $\mathcal{M}(i+1,j)$, there must be some factor summand F' of $\mathcal{M}(i+1,j)$ so that $k, k' \in Q_{F'}$. Thus the Q_F must be eventually stable, and thus so must be the number of free group factor summands in the sequence of $\mathcal{M}(i,j)$. Without loss of generality, by starting the sequence $A(i)$ and $B(j)$ far enough along, we assume the number of interpolated free group factor summands of $\mathcal{M}(i,j)$ is constant in i, j, as is the family $\{Q_F\}_F$.

Consider the embedding $\mathcal{M}(i,j) \to \mathcal{M}(i+1,j)$, which is induced by a simple step from $A_d(i)$ to $A_d(i+1)$. We will show that the inclusion of $\mathcal{M}(i,j) \to \mathcal{M}(i+1,j)$ is substandard.

First assume it is a simple step of the first kind, where we make a copies of a matrix algebra M_{n_ℓ}, $\ell \in I_{A_d(i)}$. If there is a factor summand F with $k \in Q_F$ so that $p_k^D M_{n_\ell} p_k^D \neq 0$, then $M_{n_\ell} \subseteq F$. Then, as in the previous proposition, Lemma 43 shows this is a standard embedding.

Note that if there is no such F, then M_{n_ℓ} is the corner of a matrix algebra factor summand of $\mathcal{M}(i,j)$, in which case we end up making copies of this in $\mathcal{M}(i+1,j)$.

Assume instead it is a simple step of the second kind, adding a connector between M_{n_ℓ} and $M_{n_{\ell'}}$, for some $\ell, \ell' \in I_{A_d(i)}$. If there are F, F' factor summands of $\mathcal{M}(i,j)$ which contain M_{n_ℓ} and $M_{n_{\ell'}}$ respectively, then since the number of interpolated free group factor summands does not change, $F = F'$, and as in the previous proposition, Lemma 42 shows that the inclusion of F is standard

If neither M_{n_ℓ} nor $M_{n_{\ell'}}$ is contained in an interpolated free group factor summand of $\mathcal{M}(i,j)$, then these are corners of factor summands M_m and $M_{m'}$ in $\mathcal{M}(i,j)$, which are embedded in a factor summand $M_{m'+m}$ in $\mathcal{M}(i+1,j)$.

Finally if only one of M_{n_ℓ} or $M_{n_{\ell'}}$ is contained in an interpolated free group factor summand, say M_{n_ℓ}, then either we can apply Lemma 34 again (if minimal projections in $M_{n_{\ell'}}$ are not minimal in $\mathcal{M}(i,j)$) to see that the inclusion is substandard, or it is just a dilation (if minimal projections in $M_{n_{\ell'}}$ are minimal in $\mathcal{M}(i,j)$) and therefore substandard.

Thus the inclusion of $\mathcal{M}(i,j)$ into $\mathcal{M}(i+1,j)$ is substandard. Thus by Proposition 21 the inductive limits of the interpolated free group factors are interpolated free group factors.

If $P_{\mathcal{M}}(i,j)$ is the projection onto the hyperfinite and matrix portion of $\mathcal{M}(i,j)$ then $P_{\mathcal{M}}(i,j) \geq P_{\mathcal{M}}(i',j')$ if $(i,j) \leq (i',j')$. Taking the inductive limit of those, we get a hyperfinite algebra.

As before, since $\mathrm{fdim}(M(i,j)) = \mathrm{fdim}(A(i)) + \mathrm{fdim}(B(j)) - \mathrm{fdim}(D)$, taking the limit we see $\mathrm{fdim}(A *_D B) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$. □

Corollary 49. *If A and B are hyperfinite von Neumann algebras with finite dimensional abelian von Neumann subalgebra D, so that $G_D^{A,B}$ is connected, and so that any minimal projection p in A or B, with $p \leq p_k^D$ a minimal projection in D $\tau(p) < \frac{1}{2}\tau(p_k^D)$, then $A *_D B = L(F_s)$ $s = fdim(A) + fdim(B) - fdim(D)$.*

Proof. Based on the condition $\tau(p) \leq \frac{1}{2}\tau(p_k^D)$, we know that there is no matrix algebra component for any of the $\mathcal{M}(i,j)$, and consequently none in the inductive limit. □

Definition 50. Define $\mathcal{R}_2$ to be the set of finite von Neumann algebras which are the direct sum of a finite number of interpolated free group factors and a hyperfinite von Neumann algebra. Note $\mathcal{R}_2$ strictly contains the set $\mathcal{R}_1$ from Definition 39.

Theorem 51. *For $A, B \in \mathcal{R}_2$ with finite dimensional subalgebra D, $\mathcal{M} = A *_D B$ is in $\mathcal{R}_2$. Furthermore $fdim(\mathcal{M}) = fdim(A) + fdim(B) - fdim(D)$.*

Proof. This proof is almost identical to that of Theorem 4.4 in [Dyk11] (Theorem 40 here).

By Lemma 36, without loss of generality we assume D is abelian.

We proceed by induction on the total number of interpolated free group factor summands in A and B. The base case, where there are none, is Theorem 48.

For the inductive step we assume that A contains at least one interpolated free group factor summand, F. Let p be the central support of F in A and let $\bar{p}$ be the central support of p in $\mathcal{M}$. Let $\underline{A} = pD \oplus (1-p)A$ and let $\underline{\mathcal{M}} = \underline{A} *_D B$. Then by Lemma 31, we know $p\mathcal{M}p \cong F *_{pD} (p\underline{\mathcal{M}}p)$. Note $(pD \oplus (1-p)A)$ is in $\mathcal{R}_2$ and has one fewer interpolated free group factor summand than A. Thus, by the induction hypothesis $\underline{\mathcal{M}} \in \mathcal{R}_2$. Then since F is an interpolated free group factor and $p\underline{\mathcal{M}}p$ is in $\mathcal{R}_2$, Theorem 30 tells us $p\mathcal{M}p$ is an interpolated free group factor. Then note $\mathcal{M} \cong \underline{\mathcal{M}}(1-\bar{p}) \oplus (\bar{p}\mathcal{M})$. We know $\underline{\mathcal{M}}(1-\bar{p}) \in \mathcal{R}_2$ and since $p\mathcal{M}p$ is an interpolated free group factor, so is $\bar{p}\mathcal{M}$. Thus $\mathcal{M} \in \mathcal{R}_2$.

By the induction hypothesis $\mathrm{fdim}(\underline{\mathcal{M}}) = \mathrm{fdim}(\underline{A}) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$. Let $t = \tau(\bar{p})$. Since $\mathcal{M} \cong \underline{\mathcal{M}}(1-\bar{p}) \oplus (\bar{p}\mathcal{M})$ we see $\mathrm{fdimC}_1(\mathcal{M}) = \mathrm{fdimC}_{1-t}((1-\bar{p})\underline{\mathcal{M}}) + \mathrm{fdimC}_t(\bar{p}\mathcal{M})$. By the properties of the free dimension contribution $\mathrm{fdimC}_t(\bar{p}\mathcal{M}) = \mathrm{fdimC}_{\tau(p)}(p\mathcal{M}p)$. Since $p\mathcal{M}p = F *_{pD} (p\underline{\mathcal{M}}p)$, we know $\mathrm{fdimC}_{\tau(p)}(p\mathcal{M}p) = \mathrm{fdimC}_{\tau p}(F) + \mathrm{fdimC}_{\tau p}(p\underline{\mathcal{M}}p) - \mathrm{fdimC}_{\tau(p)}(pD)$. Note $\mathrm{fdimC}_t(\bar{p}\underline{\mathcal{M}}\bar{p}) = \mathrm{fdimC}_{\tau(p)}(p\underline{\mathcal{M}}p)$ and thus

$$\mathrm{fdimC}_1(\mathcal{M}) = \mathrm{fdimC}_{1-t}((1-\bar{p}\mathcal{M}) + \mathrm{fdimC}_t(\bar{p}\underline{\mathcal{M}}\bar{p}) + \mathrm{fdimC}_{\tau(p)}(F) - \mathrm{fdimC}_{\tau(p)}(pD)$$

$$= \mathrm{fdimC}_1(\underline{\mathcal{M}}) + \mathrm{fdimC}_{\tau(p)}(F) - \mathrm{fdimC}_{\tau(p)}(pD)$$

Since

$$\mathrm{fdimC}_1(A) - \mathrm{fdimC}_1(\underline{A}) = +\mathrm{fdimC}_{\tau(p)}(F) - \mathrm{fdimC}_{\tau(p)}(pD),$$

we see $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$.

$\square$

Remark. Like in Theorem 40, if none of the interpolated free group factor summands in A or B are $L(F_\infty)$ then none in the result will be. This can be seen from the free dimension calculation, since $\mathrm{fdim}(A)$ and $\mathrm{fdim}(B)$ have to be finite so does $\mathrm{fdim}(\mathcal{M})$.

C. Examples

Example 52. The simplest case is where both algebras are diffuse hyperfinite algebras.

$$\mathcal{M} = L^\infty(\mu_1) \otimes R *_{\underset{\alpha}{\mathbb{C}} \oplus \underset{1-\alpha}{\mathbb{C}}} L^\infty(\mu_2) \otimes R.$$

In this case, as long as the graph is connected, the embedding of D does not matter, nor do the particular diffuse algebras. Since there are no minimal projections in A or B, we know that $\mathcal{M}$ is a single interpolated free group factor. We also know $\mathrm{fdim}(\mathcal{M}) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D) = 1 + 1 - (1 - \alpha^2 - (1-\alpha)^2)$, Thus $\mathcal{M} = L(F_{1+\alpha^2+(1-\alpha)^2})$.

Example 53. Next we look at an example where we get the direct sum of two free group factors, despite having connected graph. Consider:

$$A = \overset{p}{R} \oplus \underset{1/8}{M_2} \oplus \underset{1/8}{M_2} \oplus \overset{q}{R}$$

$$B = \underset{1/8}{\mathbb{C}} \oplus \underset{1/8}{M_2} \oplus \underset{1/4}{\mathbb{C}} \oplus \underset{1/8}{M_2} \oplus \underset{1/8}{\mathbb{C}}$$

$$D = \overset{p_1}{\underset{1/4}{\mathbb{C}}} \oplus \overset{p_2}{\underset{1/8}{\mathbb{C}}} \oplus \overset{p_3}{\underset{1/4}{\mathbb{C}}} \oplus \overset{p_4}{\underset{1/8}{\mathbb{C}}} \oplus \overset{p_5}{\underset{1/4}{\mathbb{C}}}$$

Where $\tau(p) = \tau(q) = 1/4$.

$$p_1 = \left(I_R, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0_R \right) \in A, \left(1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0 \right) \in B$$

$$p_2 = \left(0_R, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0_R\right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0\right) \in B$$

$$p_3 = \left(0_R, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0_R\right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0\right) \in B$$

$$p_4 = \left(0_R, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0_R\right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0\right) \in B$$

$$p_5 = \left(0_R, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, I_R\right) \in A, \left(0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 1\right) \in B$$

We use $A(i) = M_i \oplus M_2 \oplus M_2 \oplus M_i$ and $B = B_a = B(j), \forall j$. Applying Theorem 37, we see $\mathcal{M}(i,j) = L(F_{\frac{9}{8}-\frac{1}{4i^2}}) \oplus L(F_{\frac{9}{8}-\frac{1}{4i^2}})$. Thus as $i \to \infty$, $\mathcal{M} = A *_D B = L(F_{\frac{9}{8}}) \oplus L(F_{\frac{9}{8}})$. Checking, we see $\mathrm{fdim}(A) = \frac{31}{32}$, $\mathrm{fdim}(B) = \frac{7}{8}$ and $\mathrm{fdim}(D) = \frac{25}{32}$, this tells us that $\mathrm{fdim}(\mathcal{M}) = \frac{17}{16}$ which matches our result.

Example 54. This next example is a hyperfinite algebra despite having connected graph, no minimal projections of D minimal in A or B, and arbitrarily large finite dimension. Let $A = B = \underset{1/2n}{M_n} \oplus \underset{1/2n}{M_n}$ and $D = M_n(D)$, where $E^A_D((x,y)) = E^B_D((x,y)) = x + y$. Reducing to abelian D we apply Theorem 24, to get $e^D_{11}(A *_D B)e^D_{11} = L^\infty([0,\pi/2]) \otimes M_2$. Thus $A *_D B = L^\infty([0,\pi/2]) \otimes M_{2n}$

Example 55. Unlike in the multimatrix case (with finite dimensional D), we can actually end up with II_1 hyperfinite algebras. Let $A = (L^\infty[0,1] \otimes R) \oplus \underset{1/3}{M_2}$ and $B = \underset{1/3}{M_2} \oplus \underset{1/3}{\mathbb{C}}$, and $D = \overset{p_1}{\underset{1/3}{\mathbb{C}}} \oplus \overset{p_2}{\underset{1/3}{\mathbb{C}}} \oplus \overset{p_3}{\underset{1/3}{\mathbb{C}}}$. $p_1 = (I_{L^\infty \otimes R}, 0) \in A$ and $(e_{11}, 0) \in B$, $p_2 = (0, e_{11}) \in A$ and $(e_{22}, 0) \in B$, and $p_3 = (0, e_{22}) \in A$, and $(0,1) \in B$.

Then we set $A(i) = \underset{1/(3i^2)}{\mathbb{C}^i \otimes M_i} \oplus \underset{1/3}{M_2}$, and $B(j) = B$, for all j. Then $\mathcal{M}(i,j) = \underset{1/(3i^2)}{\mathbb{C}^i \otimes M_{3i}}$. Taking the inductive limit, this is $L^\infty \otimes R$.

Example 56. In this case each $\mathcal{M}(i,j)$ has matrix factor summands, but the limit does not. Note this can only happen if at least one of the minimal projections of D is minimal in A or B. Let $A = R$ and $B = \underset{\alpha/2}{\overset{p_1}{M_2}} \oplus \underset{1-\alpha}{\overset{p_2}{\mathbb{C}}}$ and $D = \underset{\alpha}{\overset{p_1}{\mathbb{C}}} \oplus \underset{1-\alpha}{\overset{p_2}{\mathbb{C}}}$, with α irrational. Then we can set $A(i) = \underset{1/2^i}{M_{2^i-1}} \oplus \underset{a}{\overset{q_1}{\mathbb{C}}} \oplus \underset{b}{\overset{q_2}{\mathbb{C}}}$, where $a + b = 1/2^i$, $q_1 \leq p_1$ and $q_2 \leq p_2$. $B(j) = B$ for all j. Then $\mathcal{M}(i, j) = L(F_s) \oplus \underset{b}{\mathbb{C}}$ for some s. Noting $b \to 0$ as $i \to \infty$, so $\mathcal{M} = L(F_s)$, where using the free dimension formula we can determine that $s = 1 + (3\alpha)/4)$.

CHAPTER V

AMALGAMATION OVER INFINITE DIMENSIONAL DISCRETE ALGEBRAS

In this chapter we examine amalgamation over a possibly infinite dimensional multimatrix algebra, instead of restricting to finite dimensions.

A. Hyperfinite Algebras

We start by adapting Lemma 41 to the infinite dimensional case:

Lemma 57. *Let A be a hyperfinite von Neumann algebra with subalgebra $D = \bigoplus_{n\in N} \underset{t_n}{\overset{p_n}{\mathbb{C}}}$, for some countable index set N. Then there exists a chain of multimatrix subalgebras A_j, so that $D \subseteq A_1 \subseteq A_2 \dots$ and $\cup_{j=1}^{\infty} A_j$ is dense in A.*

Proof. As before we divide A into a type I part and a type II part and approximate them separately. First assume A is type I, and thus of the form $B \oplus (C \otimes L^\infty([0,1]))$, where B and C are multimatrix algebras. We can choose a representation so that every p_n, $n \in N$, is composed of diagonal matrices in B and C and characteristic functions in $L^\infty([0,1])$. Let $C = \oplus_{k\in K} C_k$ where each C_k is a matrix algebra. Then for each k in K we can choose a partition P_k of $[0,1]$ into a countable number of measurable subsets so that for every $S \in P_k$ and $e_{j,j}^{(k)} \in C_k$, $e_{j,j}^{(k)} \otimes \chi_S \leq p_n$ for some $n \in N$. Then we set $A_1 = B \oplus \bigoplus_{k\in K} (C_k \otimes L^\infty(P_k))$, and note $D \subseteq A_1$. We can then define each A_j by further refining P_k to get the desired sequence.

For the type II case, as in Lemma 41, we note that our algebra is isomorphic to $L^\infty(X,\mu) \otimes R$ for some measure μ. As before we pick subalgebras M_{2^k} of R so that $\cup_{k=1}^{\infty} M_{2^k}$ is dense in R, and denote the standard basis elements of M_{2^k} by $e_{i,j}^{(k)}$, and use the inclusion $M_{2^k} \to M_{2^{k+1}}$ where $e_{i,j}^{(k)} \to e_{2i-1,2j-1}^{(k+1)} + e_{2i,2j}^{(k+1)}$.

Let $f_n \in L^\infty(X,\mu)$ be the centre-valued trace of p_n in A, for all $n \in N$. Then let $S_{n,k} = f_n^{-1}\left(\cup_{\ell=1}^{2^{k-1}} \left[\frac{2\ell-1}{2^k}, \frac{2\ell}{2^k}\right)\right)$ for $k \geq 1$ and let $S_{n,0} = f_n^{-1}(\{1\})$. Then note $f_n = \sum_{k=0}^\infty \frac{1}{2^k}\chi_{S_{n,k}}$.

Now define $P_{0,k}$ to be a countable partition of X so that for every $n \in N$, $S_{n,k}$ with non-zero measure is the union of sets in $P_{0,k}$, and so that $P_{0,k}$ refines $P_{0,k-1}$. For each $S \in P_{0,k}$, let $s_{S,k}$ be the number of $n \in N$ such that $S \subseteq S_{n,k}$, and let $r_{S,k} = s_{S,k} + 2r_{S',k-1}$ where $S \subseteq S' \in P_{0,k-1}$, (for convenience we define $P_{0,-1} = \{X\}$ and $r_{X,-1} = 0$).

Then define A_0 to be the multimatrix algebra spanned by the basis

$$B_0 = \{e_{i,j}^{(k)} \otimes \chi_S | 0 \leq k, S \in P_{0,k}, r_{S',k-1} < i,j \leq r_{S,k}, S \subseteq S' \in P_{0,k-1}\}.$$

Note then for any $x \in X$ (except possibly on a set of measure zero),

$$\sum_{\substack{e_{i,i}^{(k)} \otimes \chi_S \in B_0, \\ x \in S}} e_{i,i}^{(k)} = I_R.$$

To check this, consider the trace of this sum. This is bounded above by $\sum_{n=1}^\infty f_n(x) = 1$, and for any $\epsilon > 0$ we can find an n_0 so that $\sum_{n=1}^{n_0} f_n(x) > 1-\epsilon$ and a k_0 so that $2^{-k_0}n_0 < \epsilon$. Thus the sum of $\{\tau(e_{i,i}^{(k)}) = 2^{-k} | e_{i,i}^{(k)} \otimes \chi_S \in B_0, x \in S, k \leq k_0\}$ is at least $1-2\epsilon$. It follows from our choice of $r_{S,k}$ that the diagonal basis elements are orthogonal. It is also clear that this algebra contains a set of orthogonal projections p_n' with centre-valued traces f_n for $n \in N$. These are then equivalent to the p_n in D, and thus without loss of generality we can identify them with D.

To define B_m, we first define $P_{m,k}$ to be a refinement of $P_{m-1,k}$ such for any $S \in P_{m,k}$ either $0 < \mu(S) \leq 2^{-m}$ or S is a single atom. We define $r_{S,k}$ and $s_{S,k}$ in the same way (note for $S \in P_{m,k}$ if there exists an $S' \in P_{m',k}$ with $S \subseteq S'$ and $m > m'$ then $r_{S,k} = r_{S',k}$ and $s_{S,k} = s_{S',k}$ so this is well defined, and we do not need to add an index m).

Then let A_m be the multimatrix algebra spanned by

$$B_m = \{e_{i,j}^{(k)} \otimes \chi_S | m < k, S \in P_{m,k}, r_{S',k-1} < i,j \leq r_{S,k}, S \subseteq S' \in P_{m,k-1},$$
$$\text{or if } k = m, S \in P_{m,k}, 1 \leq i,j \leq r_{S,m}\}.$$

Recalling the inclusion of $M_{2^{m-1}}$ into M_{2^m} and that each $P_{m,k}$ is a refinement of $P_{m-1,k}$, we see $A_{m-1} \subseteq A_m$.

We have established that for any $x \in X$ (except possibly on a set of measure zero) as m goes to infinity the sum of $\{\tau(e_{i,i}^{(k)}) = 2^{-k} | e_{i,i}^{(k)} \otimes \chi_S \in B_0, x \in S, k \leq m\}$ goes to 1. Since for A_m there is a set S containing x with measure less than $\max\{\mu(\{x\}), 2^{-m}\}$ and matrix subalgebra $M_{m'} \otimes \chi_S \subseteq M_{2^m} \otimes \chi_S$, whose trace is $\mu(S)$ times that sum. Thus this is dense in $\cup_m M_{2^m} \otimes L^\infty(X,\mu)$ and thus dense in A

□

Lemma 58. *For two multimatrix algebras $\mathcal{N}$ and $\mathcal{M}$ and a trace preserving inclusion $\phi : \mathcal{N} \to \mathcal{M}$, then ϕ can be written as a composition of a (possibly countably infinite) sequence of simple steps.*

Proof. The proof is identical to Lemma 45, except that since we have an infinite number of matrix algebras, each of which can be split into an infinite number of matrix algebras, we may need an infinite number of steps. □

Theorem 59. *Let A and B be hyperfinite von Neumann algebras, with multimatrix subalgebra D and trace preserving conditional expectations E_D^A and E_D^B onto D. Then $A *_D B$ is of the form $H \oplus \bigoplus_{i \in I} F_i$ where H is a hyperfinite algebra and the F_i are interpolated free group factors and I is countable. Furthermore fdim$(A *_D B) =$ fdim$(A) +$ fdim$(B) -$ fdim(D).*

Proof. As usual, by Lemma 36 we can assume without loss of generality that D is abelian, and since if D is finite dimensional we can apply Theorem 48 we may also assume $D = \bigoplus_{k=1}^{\infty} \underset{t_k^D}{\overset{p_k^D}{\mathbb{C}}}$.

Use Lemma 57 to define the sequences A_i and B_j to be chains of subalgebras of A in B, respectively, containing D. Note in the construction from Lemma 57, the atomic parts of A an B are included in every A_i and B_j. Let $q_k^D = \sum_{k'=1}^{k} p_{k'}^D$. Then let

$$\mathcal{M}(i,j,k) = \left(q_k^D A_j q_k^D *_{q_k^D D q_k^D} q_k^D B_j q_k^D\right) \oplus (1-q_k^D)D(1-q_k^D),$$

and let $\mathcal{N}(i,j,k)$ be

$$\left(q_{k-1}^D A_j q_{k-1}^D *_{q_{k-1}^D D q_{k-1}^D} q_{k-1}^D B_j q_{k-1}^D\right) \oplus \left(p_k^D A_i p_k^D *_{p_k^D D p_k^D} p_k^D B_j p_k^D\right) \oplus (1-q_k^D)D(1-q_k^D).$$

For fixed k, the sequence $\mathcal{M}(i,j,k)$ (or more precisely $q_k^D\mathcal{M}(i,j,k)q_k^D$), is an approximating sequence for the amalgamated free product of hyperfinite von Neumann algebras over $q_k^D D$ (which is finite dimensional), as in Theorem 48. Thus we can choose i_k and j_k sufficiently large to be the start of the sequence in the proof of that theorem. In particular, for any minimal projection $p \in A_{i_k}$ (resp B_{j_k}) from the diffuse part of A (resp. B), so that $p \leq p_{k'}^D$ with $k' \leq k$ then $\tau(p) < t_{k'}^D - \tau(p')$ for any minimal projection $p' \in B$ (resp A) with $p' \leq p_{k'}^D$, unless $p_{k'}^D$ is minimal in B (resp. A), and also so that the number of interpolated free group factors in $\mathcal{M}(i,j,k)$ is stable for all $i \geq i_k$ and $j \geq j_k$.

Our goal will be to show that the the inclusions of the sequence

$$\mathcal{M}(i_2,j_2,1) \to \mathcal{N}(i_2,j_2,2) \to \mathcal{M}(i_2,j_2,2) \to \mathcal{M}(i_3,j_3,2) \to \mathcal{N}(i_3,j_3,3) \to \dots$$

are substandard.

First examine the steps of the form $\mathcal{M}(i_k,j_k,k) \to \mathcal{M}(i_{k+1},j_{k+1},k)$. Using

Lemma 58, we break this up into a sequence of inclusions induced by simple steps in the subalgebras of A or B, so we need only show that this type of inclusion is substandard. Consider the inclusion $\mathcal{M}(i,j,k) \to \mathcal{M}(i+1,j,k)$ with $i_k \leq i < i_{k+1}$ and $j_k \leq j \leq j_{k+1}$ where $A_i \to A_{i+1}$ is a simple step (note this may not work precisely with our original indexing, but this does not affect the proof and eases notation).

First assume the inclusion $A_i \to A_{i+1}$ is a simple step of the first kind, which makes copies of a summand, M_n, of A_i. As in the proof of Theorem 48, either M_n is contained in some interpolated free group factor summand, or it is the corner of some matrix factor summand. In the former case we apply Lemma 43 to show this is a standard embedding. In the latter we see the matrix factor is similarly copied.

Alternately, if the inclusion $A_i \to A_{i+1}$ is a simple step of the second kind, then we are adding a connector v between two matrix algebras in A_i. Again we proceed exactly as in Theorem 48. If both vv^* and v^*v are in an interpolated free group factor we apply Lemma 42 to show this is a standard embedding. If only one of vv^* and v^*v is in an interpolated free group factor we apply Lemma 34 to show that this is a substandard embedding. If neither vv^* nor v^*v are in interpolated free group factors, then they are both in matrix algebras, then these are embedded in another matrix algebra.

Thus applying Proposition 21 to this sequence of inclusions we see then that any interpolated free group factor summand in $\mathcal{M}(i_{k-1}, j_{k-1}, k)$, is embedded in another in $\mathcal{M}(i_k, j_k, k)$ by a substandard embedding, and so this inclusion is substandard.

Next we examine the inclusions of the form $\mathcal{N}(i_k, j_k, k) \to \mathcal{M}(i_k, j_k, k)$. This is similar to the proof of Theorem 37.

Let $q_k A_{i_k} q_k = \bigoplus_{\ell \in I_{A_{i_k}}} \overset{p_\ell}{M_{n_\ell}}$ and $q_k B_{j_k} q_k = \bigoplus_{\ell \in I_{B_{i_k}}} \overset{p_\ell}{M_{n_\ell}}$. Now for each $\ell \in I_{A_{i_k}} \cup I_{B_{j_k}}$, if p_ℓ is not orthogonal to either q_{k-1} or p_k, then choose a $v_\ell \in M_{n_\ell}$ which is a partial isometry so that $v_\ell v_\ell^* \leq p_k$ and $v_\ell^* v_\ell \leq q_{k-1}$. Let V be the set of these v_ℓ. Then note that $\mathcal{N}(i_k, j_k, k) \cup V$ generates $\mathcal{M}(i_k, j_k, k)$. Let $\mathcal{N}(i_k, j_k, k, V') = vN(\mathcal{N}(i_k, j_k, k) \cup V')$ for $V' \subseteq V$. We show that for $v \in V$ and $V' \subseteq V \backslash \{v\}$, the inclusion $\mathcal{N}(i_k, j_k, k, V') \to \mathcal{N}(i_k, j_k, k, V' \cup \{v\})$ is substandard.

Let $q = vv^* + v^*v$, and examine $q\mathcal{N}(i_k, j_k, k, V')q$, and the algebra M_2 generated by v. As in the proofs of Theorem 37 and Theorem 48, we see that

$$q\mathcal{N}(i_k, j_k, k, V')q \underset{\underset{vv^*}{\mathbb{C}} \oplus \underset{v^*v}{\mathbb{C}}}{*} M_2 = q\mathcal{N}(i_k, j_k, k, V' \cup \{v\})q.$$

If either of vv^* or v^*v are minimal and central in $q\mathcal{N}(i_k, j_k, k, V')q$, (without loss of generality assume vv^*), then vv^* is a minimal projection in a matrix algebra summand M_n in $\mathcal{N}(i_k, j_k, k, V')$. Any matrix algebra summand $M_{m'}$ under v^*v in $q\mathcal{N}(i_k, j_k, k, V')q$ must be part of a matrix algebra summand M_m in $\mathcal{N}(i_k, j_k, k, V')$. Then the inclusion of M_m in $\mathcal{N}(i_k, j_k, k, V' \cup \{v\})$ maps it to the corner of a matrix summand $M_{m+nm'}$. Any interpolated free group factor under v^*v, is included into a dilation of itself, which is a substandard embedding. Finally if there is a diffuse hyperfinite algebra under v^*v then it is included in a diffuse hyperfinite algebra.

If neither vv^* nor v^*v is minimal and central, then we can apply Lemma 34. This shows us that the inclusion of $q\mathcal{N}(i_k, j_k, k, V')q$ into $q\mathcal{N}(i_k, j_k, k, V' \cup \{v\})q$ is substandard, and thus so is the inclusion of $\mathcal{N}(i_k, j_k, k, V')$ into $\mathcal{N}(i_k, j_k, k, V' \cup \{v\})$. Thus, applying Proposition 21, so is the inclusion $\mathcal{N}(i_k, j_k, k) \to \mathcal{M}(i_k, j_k, k)$.

In the final case, $\mathcal{M}(i_k, j_k, k-1) \to \mathcal{N}(i_k, j_k, k)$, aside from the inclusion of the algebra spanned by p_k^D, all inclusions are the identity.

Thus the inclusions in the chain,

$$\mathcal{M}(i_2, j_2, 1) \to \mathcal{N}(i_2, j_2, 2) \to \mathcal{M}(i_2, j_2, 2) \to \mathcal{M}(i_3, j_3, 2) \to \mathcal{N}(i_3, j_3, 3) \to \ldots,$$

are all substandard. At all stages this is the direct sum of a finite number of interpolated free group factors and a hyperfinite algebra. Thus, applying Proposition 21 once more, the inductive limit is a countable direct sum of interpolated free group factors and a hyperfinite algebra.

From this it is clear that $\mathrm{fdim}(A *_D B) = \lim_{k\to\infty} \mathrm{fdim}(\mathcal{M}(i_k, j_k, k))$. Choose $k \in I_D$ and denote $A_{i_k} = \bigoplus_{\ell\in I_A} \underset{t_\ell}{\overset{p_\ell}{M_{n_\ell}}}$. Then

$$\mathrm{fdimC}_{\tau(q_k)}(q_k A_{i_k} q_k) = -\sum_{\ell\in I_A, p_\ell q_k \neq 0} t_\ell^2 - \sum_{k'>k} (t_k^D)^2,$$

and

$$\mathrm{fdim}(A_{i_k}) = 1 - \sum_{\ell\in I_A} t_\ell^2.$$

For every $\ell \in I_D$ so that $p_\ell q_k = 0$, there is a minimal projection $p'_\ell \in M_{n_\ell}$ less than some $p^D_{k'}$ with $k' > k$. This tells us that

$$\sum_{\ell\in I_D, p_\ell q_k = 0} t_\ell^2 < \sum_{k'>k} (t_{k'}^D)^2.$$

Thus

$$\left|\mathrm{fdimC}_{\tau(q_k)}(q_k A_{i_k} q_k) - \mathrm{fdimC}_1(A_{i_k})\right| < \sum_{k'>k} (t_{k'}^D)^2 = \mathrm{fdimC}_{\tau(1-q_k)}((1-q_k)D(1-q_k)).$$

The same calculation works for B_{j_k}. Then note by Theorem 48

$$\begin{aligned}
&\mathrm{fdimC}_{\tau(q_k)}(q_k A_{i_k} q_k *_{q_k D q_k} q_k B_{j_k q_k}) \\
&\qquad = \tau(q_k)^2 \left(\mathrm{fdim}(q_k A_{i_k} q_k) + \mathrm{fdim}(q_k B_{j_k} q_k) - \mathrm{fdim}(q_k D q_k) - 1\right) \\
&\qquad\qquad = \mathrm{fdimC}_{\tau(q_k)}(q_k A_{i_k} q_k) + \mathrm{fdimC}_{\tau(q_k)}(q_k B_{j_k} q_k) - \mathrm{fdimC}_{\tau(q_k)}(q_k D q_k).
\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{fdimC}_1(\mathcal{M}(i_k,j_k,k)) &= \mathrm{fdimC}_{\tau(q_k)}(q_kA_{i_k}q_k)+\mathrm{fdimC}_{\tau(q_k)}(q_kB_{j_k}q_k)\\ &\quad -\mathrm{fdimC}_{\tau(q_k)}(q_kDq_k)+\mathrm{fdimC}_{\tau(1-q_k)}((1-q_k)D(1-q_k)).\end{aligned}$$

Then since $\lim_{k\to\infty}\mathrm{fdimC}_{\tau(1-q_k)}((1-q_k)D(1-q_k))=0$, we see

$$\begin{aligned}\lim_{k\to\infty}\mathrm{fdimC}_1(\mathcal{M}(i_k,j_k,k)) &= \lim_{k\to\infty}\mathrm{fdimC}_1A_{i_k}+\lim_{k\to\infty}\mathrm{fdimC}_1B_{i_k}-\mathrm{fdimC}_1(D)\\ &= \mathrm{fdimC}_1A+\mathrm{fdimC}_1B-\mathrm{fdimC}_1(D)\end{aligned}$$

Thus $\mathrm{fdim}(A*_D B)=\mathrm{fdim}(A)+\mathrm{fdim}(B)-\mathrm{fdim}(D)$.

□

Example 60. The following example shows that the product can be an infinite direct sum of interpolated free group factors, even with connected graph. Let:

$$A=\underset{\frac{1}{3}}{\overset{p_0^A}{\mathbb{C}}}\oplus\bigoplus_{i=1}^{\infty}\underset{\frac{1}{3(2^i)}}{\overset{p_i^A}{M_2}},$$

$$B=\bigoplus_{i=1}^{\infty}\left(\overset{p_{2i-1}^B}{R}\oplus\underset{\frac{1}{4^i}}{\overset{p_{2i}^B}{\mathbb{C}}}\right),$$

$$D=\bigoplus_{i=1}^{\infty}\underset{\frac{1}{2^i}}{\overset{p_i^D}{\mathbb{C}}}.$$

we use the inclusion of D so that $p_i^B=p_i^D$ and one of the minimal projections of p_i^A is under p_i^D and the other is under p_{i+1}^D (except for p_0^A which is entirely under p_1^D). Define $\mathcal{N}(k)=p_k^DAp_k^D*p_k^DBp_k^D$, we see:

$$\mathcal{N}(k)=\left(\underset{\frac{1}{3(2^{k-1})}}{\mathbb{C}}\oplus\underset{\frac{1}{3(2^k)}}{\mathbb{C}}\right)*R=L(F_{\frac{13}{9}}),$$

if k is odd and

$$\mathcal{N}(k) = \left(\underset{\frac{1}{3(2^{k-1})}}{\mathbb{C}} \oplus \underset{\frac{1}{3(2^{k})}}{\mathbb{C}} \right) * \mathbb{C} = \left(\underset{\frac{1}{3(2^{k-1})}}{\mathbb{C}} \oplus \underset{\frac{1}{3(2^{k})}}{\mathbb{C}} \right),$$

if k is even.

Then to get $A *_D B$ we must adjoin a connector from each of the M_2 in A. In each case these connectors connect an $\mathcal{N}(k)$ to $\mathcal{N}(k+1)$, and thus an even one to an odd one. This results in a dilation of the odd one. Thus for every odd $k \geq 3$ we get a copy $L(F_{\frac{10}{9}})$ with trace $\frac{1}{2^{k-1}}$, and for $k = 1$ we are left with $L(F_{\frac{5}{4}})$ with trace $\frac{2}{3}$. So ultimately,

$$A *_D B = L(F_{\frac{5}{4}}) \oplus \bigoplus_{k=1}^{\infty} L(F_{\frac{10}{9}}).$$

Then calculating the free dimension of $A *_D B$ we get $1 + \left(\frac{2}{3}\right)^2 \frac{1}{4} + \sum_{k=1}^{\infty} \left(\frac{1}{2^{2k}}\right)^2 \frac{1}{9} = \frac{151}{135}$. Note $\mathrm{fdim}(A) = 1 - \frac{1}{9} - \sum_{i=1}^{\infty} \frac{1}{(3(2^i))^2} = \frac{23}{27}$, $\mathrm{fdim}(B) = 1 - \sum_{i=1}^{\infty} \frac{1}{(4^i)^2} = \frac{14}{15}$ and $\mathrm{fdim}(D) = 1 - \sum_{i=1}^{\infty} \frac{1}{(2^i)^2} = \frac{2}{3}$, and thus $\mathrm{fdim}(A *_D B) = \mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$.

B. Hyperfinite Algebras and Interpolated Free Group Factors

Definition 61. Define $\mathcal{R}_3$ to be the set of finite von Neumann algebras which are of the form $H \oplus \bigoplus_{i \in I} F_i$, where H is a hyperfinite von Neumann algebra, the F_i are interpolated free group factors, and I is countable.

Our goal for this section is to show that $\mathcal{R}_3$ is closed under amalgamated free products over multimatrix subalgebras. Note that $\mathcal{R}_3$ strictly contains $\mathcal{R}_2$ defined earlier, and is the smallest class containing the hyperfinite algebras that can be closed under this type of amalgamated free products. We continue by proving a few necessary lemmas.

The following lemma was proved as Theorem 3.1 in [BD04].

Lemma 62. *Let R and R' be hyperfinite II_1 factors. Let $D = \oplus_{i\in I}\underset{t_i}{\overset{p_i}{\mathbb{C}}}$ (I countable) be a common subalgebra of R and R'. Then $R *_D R' = L(F_{1+\sum_{i\in I} t_i^2})$, and this is generated by $R' \cup \{p_i X_i p_i\}_{i\in I}$.*

Proof. The fact that $R *_D R'$ is the interpolated free group factor mentioned follows from Theorem 59, so here we need only check that it is generated correctlly.

First assume that I is finite, and proceed by induction on the size of I. In the case $|I| = 1$, this has been established in Corollary 3.6 in [Dyk94]. Let i_0 be such that t_{i_0} is minimal. Using Theorem 2.1 (b) from [Dyk11], we define

$$\mathcal{N}_1 = vN\left((1-p_{i_0})R(1-p_{i_0}) \cup (1-p_{i_0})R'(1-p_{i_0})\right)$$

$$= (1-p_{i_0})R(1-p_{i_0}) *_{(1-p_{i_0})D} (1-p_{i_0})R'(1-p_{i_0}).$$

Then by our induction hypothesis, $\mathcal{N}_1 = L(F_{1+\sum_{i\in I, i\neq i_0}\frac{t_i^2}{(1-t_{i_0})^2}})$, generated by $(1-p_{i_0})R'(1-p_{i_0}) \cup \{p_i X_i p_i\}_{i\in I\setminus\{i_0\}}$

Then using part (c) of Theorem 2.1 in [Dyk11], we see that $r = p_{i_0}$, thus $\tilde{A} = (1-p_{i_0})R(1-p_{i_0}) \oplus \mathbb{C}p_{i_0}$, and thus $p_{i_0}\tilde{A}p_{i_0} = \mathbb{C}$. From this we see directly that for $\mathcal{N}_2 = vN(\tilde{A}\cup R')$, $\mathcal{N}_1 = (1-p_{i_0})\mathcal{N}_2(1-p_{i_0})$. Thus we write $\mathcal{N}_2 = vN(R' \cup \{p_i X_i p_i\}_{i\in I, i\neq i_0})$.

Using part (d) of Theorem 2.1 in [Dyk11], we see that since $p_{i_0}(R *_D R')p_{i_0} = p_{i_0}\mathcal{N}_2 p_{i_0} * L(\mathbb{Z})$, and that $R *_D R'$ is generated by $R' \cup \{p_i X_i p_i\}_{i\in I}$, and thus we are done.

For the infinite case, let $K = \mathbb{N}$, ordered so that $t_i \leq t_{i-1}$, and let $q_k = \sum_{i=1}^k p_i^D$. Let $\mathcal{M}_k = q_k R q_k *_{q_k D} q_k R' q_k$. The above tells us that each $\mathcal{M}_k$ is generated by $q_k R' q_k \cup \{p_i X_i p_i\}_{i=1}^k$ and that the embeddings of $\mathcal{M}_k \to \mathcal{M}_{k+1}$ are substandard, thus the generating set in the inductive limit $\mathcal{M}$ is as desired, completing the proof.

$\square$

Lemma 63. *Let A and B be interpolated free group factors, with subalgebra $D =$*

$\bigoplus_{k\in K} \underset{t_k}{\overset{p_k^D}{\mathbb{C}}}$. *Then* $\mathcal{M} = A *_D B$ *is an interpolated free group factor, fdim*$(\mathcal{M})$ = *fdim*(A) + *fdim*(B) − *fdim*(D)*, and the inclusion of* $A \to \mathcal{M}$ *is standard.*

Proof. Let p be a minimal projection in D, and choose representations of A and B of the form $A = vN(R \cup \{p_i X_i p_i\}_{i\in I})$ and $B = vN(R' \cup \{q_j X_j q_j\}_{j\in J})$, where $p_i \le p$ and $q_j \le p$ for all $i \in I, j \in J$. Then $A *_D B$ is generated by $R \cup R' \cup \{p_i X_i p_i\}_{i\in I} \cup \{q_j X_j q_j\}_{j\in J}$. From Theorem 62 we know that $R *_D R' = L(F_{2-\mathrm{fdim}(D)})$, with R embedded correctly and so $R *_D R'$ is generated by $R \cup \{p_k^D X_k p_k^D\}_{k\in K}$. We can then find $q_j' \in R$ which are unitary conjugates of q_j using unitaries in $R *_D R'$, thus $\mathcal{M}$ is generated by $R \cup \{p_k^D X_k p_k^D\}_{k\in K} \cup \{p_i X_i p_i\}_{i\in I} \cup \{q_j X_j q_j\}_{j\in J}$. Thus $\mathcal{M}$ is the desired interpolated free group factor, and the inclusion of A is standard. □

Lemma 64. *Let* $A \in \mathcal{R}_3$ *and* B *be an interpolated free group factor. Let* D *be an abelian multimatrix subalgebra of both* A *and* B*. Then* $\mathcal{M} = A *_D B$ *is an interpolated free group factor with free dimension fdim*(A) + *fdim*(B) − *fdim*(D)*. Furthermore the inclusion* $B \to \mathcal{M}$ *is standard and that of* $F_i \to \mathcal{M}$ *is substandard for any interpolated free group factor summand,* F_i*, of* A*.*

Proof. Let $A = \overset{p_0}{H} \oplus \bigoplus_{i=1}^{K} \overset{p_i}{F_i}$ for $K \in \mathbb{N} \cup \{\infty\}$ and where H is a hyperfinite algebra and each F_i is an interpolated free group factor. Let $A_0 = H \oplus \bigoplus_{i=1}^{K} p_i D$ and let $A_k = H \oplus \bigoplus_{i=1}^{k} F_i \oplus \bigoplus_{i=k+1}^{K} p_i D$. Let $\mathcal{M}_k = A_k *_D B$.

Our goal is to show that each $\mathcal{M}_k$ is an interpolated free group factor with $\mathrm{fdim}(\mathcal{M}_k) = \mathrm{fdim}(A_k) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$, and that the inclusion of $\mathcal{M}_k \to \mathcal{M}_{k+1}$ is standard.

First we show that $\mathcal{M}_0$ is an interpolated free group factor. Using Lemma 57 we can find a chain of multimatrix subalgebras of H, $\{H_j\}_{j=1}^{\infty}$ so that $p_0 D p_0 \subseteq H_1$ and $\cup_{j=1}^{\infty} H_j$ is dense in H. Now Lemma 5.9 of [Dyk95] tells us that each $\mathcal{N}_j = (H_j \oplus \bigoplus_{i=1}^{K} p_i D) *_D B$ is an interpolated free group factor with $\mathrm{fdim}(\mathcal{N}_j) = \mathrm{fdim}((H_j \oplus$

$\bigoplus_{i=1}^{K} p_i D)) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$, and the inclusion of $B \to \mathcal{N}_j$ is standard. Using Lemma 58, Lemma 43, and Lemma 42 in the usual way we see that these embeddings are standard, and thus $\mathcal{M}_0$ is an interpolated free group factor with free dimension $\mathrm{fdim}(A_0) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$, and the inclusion of $B \to \mathcal{M}_0$ is standard.

Lemma 31 tells us that $p_k \mathcal{M}_k p_k = p_k \mathcal{M}_{k-1} p_k *_{p_k D} F_k$ and that $\mathcal{M}_k$ is a factor. Thus the embedding of $F_k \to p_k \mathcal{M}_k p_k$ is standard. By Lemma 63, $p_k \mathcal{M}_k p_k$ is an interpolated free group factor and the inclusion of $p_k \mathcal{M}_{k-1} p_k$ into it is standard, and thus $\mathcal{M}_k$ is an interpolated free group factor with free dimension $\mathrm{fdim}(A_k) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$, and $\mathcal{M}_{k-1} \to \mathcal{M}_k$ is standard.

Thus the inductive limit of the $\mathcal{M}_k$, $A *_D B$, is an interpolated free group factor with free dimension $\mathrm{fdim}(A) + \mathrm{fdim}(B) - \mathrm{fdim}(D)$, the inclusion of B into it is standard, and the inclusions of $F_i \to \mathcal{M}$ are substandard. □

Theorem 65. *For $A, B \in \mathcal{R}_3$ and D a multimatrix subalgebra of them, $A *_D B \in \mathcal{R}_3$, and fdim$(A *_D B) =$ fdim$(A) +$ fdim$(B) -$ fdim(D).*

Proof. Let $A = H_A \oplus \bigoplus_{i=1}^{K_A} \overset{p_i}{F_i}$ where the F_i are interpolated free group factors and $K_A \in \mathbb{N} \cup \{\infty\}$ similarly let $B = H_B \oplus \bigoplus_{j=1}^{K_B} \overset{q_j}{F'_j}$. Then define $A_k = H_A \oplus \bigoplus_{i=1}^{k} F_i \oplus \bigoplus_{i=k+1}^{K_A} p_i D$ and define B_k similarly.

Let $\mathcal{M}(i,j) = A_i *_D B_j$, then the inductive limit of the $\mathcal{M}(i,j)$ is $A *_D B$. We claim that each $\mathcal{M}(i,j)$ is in $\mathcal{R}_3$, and that the embedding of $\mathcal{M}(i,j)$ into $\mathcal{M}(i+1,j)$ is substandard. We proceed by induction on i, j.

Theorem 59 tells us that $\mathcal{M}(0,0)$ is in $\mathcal{R}_3$, and that $\mathrm{fdim}(\mathcal{M}(0,0)) = \mathrm{fdim}(A_0) + \mathrm{fdim}(B_0) - \mathrm{fdim}(D)$. By Lemma 31, $p_i \mathcal{M}(i,j) p_i = vN(p_i \mathcal{M}(i-1,j) p_i \cup F_i) = p_i \mathcal{M}(i-1,j) p_i *_{p_i D} F_i$. Lemma 64 tells us that this is an interpolated free group factor and that the inclusion of any interpolated free group factor summand in $p_i \mathcal{M}(i-1,j) p_i \to p_i \mathcal{M}(i,j) p_i$ is substandard. This also tells us that the inclusion of $F_i \to p_i \mathcal{M}(i,j) p_i$

is standard.

Since the central support of p_i in $\mathcal{M}(i-1,j)$ and $\mathcal{M}(i,j)$ is the same (again by Lemma 31), any interpolated free group factor summand of $\mathcal{M}(i-1,j)$ which is orthogonal to p_i is identical in $\mathcal{M}(i,j)$. Thus the inclusion of $\mathcal{M}(i-1,j) \to \mathcal{M}(i,j)$ is substandard, as is the inclusion of $F_i \to \mathcal{M}(i-1,j)$.

Next note that $\mathrm{fdimC}_{\tau(q)}(q\mathcal{M}(i,j)q) = \mathrm{fdimC}_{\tau(p_i)}(p_i\mathcal{M}(i,j)p_i)$ by the properties of free dimension contribution. Note

$$\mathrm{fdimC}_{\tau(p_i)}(p_i\mathcal{M}(i,j)p_i) = \mathrm{fdimC}_{\tau(p_i)}(p_i\mathcal{M}(i-1,j)p_i) + \mathrm{fdimC}_{\tau(p_i)}(F_i) - \mathrm{fdimC}_{\tau(p_i)}(p_iD).$$

Since q is the central support of p_i in both $\mathcal{M}(i,j)$ and $\mathcal{M}(i-1,j)$ we know

$$\mathrm{fdimC}_{\tau(p_i)}p_i(\mathcal{M}(i-1,j)p_i) = \mathrm{fdimC}_{\tau(q)}(q\mathcal{M}(i-1,j)q).$$

Thus we see

$$\mathrm{fdimC}_{\tau(q)}(q\mathcal{M}(i,j)q) = \mathrm{fdimC}_{\tau(q)}(q\mathcal{M}(i-1,j)q) + \mathrm{fdimC}_{\tau(p_i)}(F_i) - \mathrm{fdimC}_{\tau(p_i)}(D).$$

And since $(1-q)\mathcal{M}(i,j)(1-q) = (1-q)\mathcal{M}(i-1,j)(1-q)$, we see

$$\mathrm{fdimC}_1(\mathcal{M}(i,j)) = \mathrm{fdimC}_1(\mathcal{M}(i-1,j)) + \mathrm{fdimC}_{\tau(p_i)}(F_i) - \mathrm{fdimC}_{\tau(p_i)}(p_iD).$$

Note $\mathrm{fdim}_{\tau(p_i)}(p_iA_ip_i) = \mathrm{fdimC}_{\tau(p_i)}(F_i)$, and $\mathrm{fdimC}_{\tau(p_i)}(p_iA_{i-1}p_i) = \mathrm{fdimC}_{\tau(p_i)}(p_iD)$. Thus

$$\mathrm{fdimC}_1(A_i) - \mathrm{fdimC}_1(A_{i-1}) = \mathrm{fdimC}_{\tau(p_i)}(F_i) - \mathrm{fdimC}_{\tau(p_i)}(p_iD).$$

Combining these we get:

$$\begin{aligned}
&\mathrm{fdimC}_1(\mathcal{M}(i,j)) \\
&\quad = \mathrm{fdimC}_1(A_{i-1}) + \mathrm{fdimC}_1(B_j) - \mathrm{fdimC}_1(D) + \mathrm{fdimC}_{\tau(p_i)}(F_i) - \mathrm{fdimC}_{\tau(p_i)}(p_iD)
\end{aligned}$$

$$= \mathrm{fdimC}_1(A_i) + \mathrm{fdimC}_1(B_j) - \mathrm{fdimC}_1(D).$$

Thus our claim is proved, and the result follows. $\square$

Unlike in Theorem 40 and Theorem 51, in $\mathcal{R}_3$ even if neither A nor B have $L(F_\infty)$ as a factor summand, $A *_D B$ may have an $L(F_\infty)$ summand.

Example 66. Let $A = \bigoplus_{i=1}^{\infty} L(\overset{p_i}{F_{4^i+1}})$ where $\tau(p_i) = \frac{1}{2^i}$ and let $B = L(F_2)$, and choose any multimatrix subalgebra D of these. Checking, we see $\mathrm{fdim}(A) = 1 + \sum_{i=1}^{\infty} 1 = \infty$. By Lemma 64 we know $A *_D B$ is an interpolated free group factor, and applying the free dimension formula we see that it must be $L(F_\infty)$.

CHAPTER VI

SUMMARY

We have described the amalgamated free product of hyperfinite von Neumann algebras over finite dimensional and multimatrix subalgebras. We have also shown the classes $\mathcal{R}_2$ and $\mathcal{R}_3$ are closed under amalgamated free products of finite dimensional and multimatrix subalgebras respectively.

Future work in this direction may look at expanding both the class of algebras we take the free product of and the subalgebras we allow. It would be interesting to allow diffuse hyperfinite subalgebras, though many of the techniques used here would not extend. Another interesting extension might be to allow the direct integral of interpolated free group factors, rather than just the countable (in $\mathcal{R}_3$) or finite (in $\mathcal{R}_2$) number.

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